DISCRETIZATION SCHEMES FOR SUBORDINATED PROCESSES

CLAUDIO ALBANESE AND ALEXEY KUZNETSOV

ABSTRACT. We introduce a new class of continuous time lattices which are suitable for local Levy and stochastic volatility processes and use them to construct numerical discretizations for the corresponding partial integro-differential equations. Transition probabilities are computed either analytically or by means of numerical linear algebra.

1. INTRODUCTION

The upsurge of interest in option pricing models based on jump processes, notably by Merton [1], Eberlein [2] and Madan et al. [3], motivated the development of suitable lattice approximation schemes. In the continuous limit, price functions solve partial integro-differential equations (PIDE's). In the context of the variance-gamma model of [4] and [3], Madan and Hirsa introduce in [5] a lattice discretization scheme for the integral kernel of the relevant PIDE. In [6], Petersdorff and Schwab propose a wavelet compression method for the resulting matrices and stability conditions appear in [7].

Lattice models for jump processes are more subtle to implement than the analogue models for diffusions. In the latter case, simple trinomial models are suitable (see Fig. 1) and transition probabilities can be set by matching the first two moments, drift and volatility. However, if the underlying process has jumps, moments higher than the second have to be fine tuned, the hopping range is not limited to nearest neighbors and scaling laws are more complex. The problem is all the more delicate for models combining jumps with state dependent local volatility which lack of translation invariance.

In this article we introduce continuous time lattice models which may be regarded as time changed versions of trinomial lattice models in the limit of a vanishing time step. The ability to take the continuous time limit hinges upon the ability to efficiently compute transition probabilities across arbitrarily long time intervals. This context is particularly suitable to model jump and stochastic volatility processes as, by not committing to a finite time step, time changes may be applied.

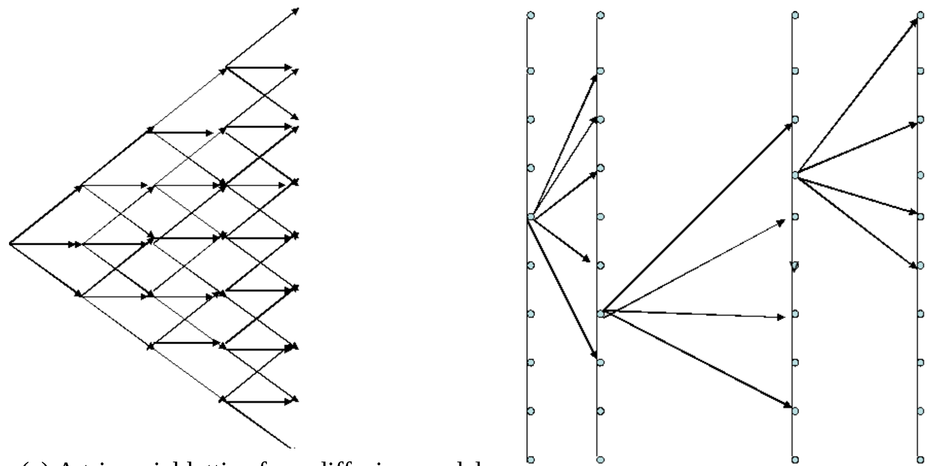
When pricing (or calibrating against) European style options with a continuous time lattice, one reduces to a single period model by computing transition probabilities from current time to maturity. A Bermudan type option or a discretely monitored barrier option would instead require the insertion of time nodes in correspondence with the exercise or monitoring dates. Continuously monitored barrier options can often be priced in one single step similarly to European options, as the boundary condition may be accommodated within the continuous time framework. A discussion of several examples is given in conclusion of this paper.

The method applies in broad generality to models for which the spectrum of the Markov generator can either be computed in analytically closed form or by means of efficient numerical linear algebra. We first discuss the case of plain Levy processes and see them as subordinated Brownian motions. Bochner subordinators are a class of monotonously increasing stochastic processes reviewed in section 3 which can be used to construct jump processes. The construction takes the premises from a Markov generator written as a second order differential operator and with a Bernstein function $\phi(\lambda)$ defining the stochastic subordinator. The Markov generator of the subordinated process is represented as the Bernstein function of the base generator using the functional calculus outlined in section 3. An approximating process is roughly defined as a continuous time Markov process $\tilde{X}^h_t$ defined on a lattice, which converges to

---

Date: September 5, 2003.

This work was written while the authors were visiting the Department of Mathematics of the National University of Singapore, whose hospitality is gratefully aknowledged. The authors were supported in part by the National Science and Engineering Council of Canada under grant RGPIN-171149 and by the Ontario Council of Graduate Studies. We thank Peter Carr, Oliver Chen, Ken Jackson and Stephan Lawi for discussions. Remaining errors are our own.



(a) A trinomial lattice for a diffusion model. Time steps are of equal length and only nearest neighbor hops occur.

(b) A lattice for a jump process. Notice the uneven time steps and long range transitions.

FIGURE 1.

the given jump process $\tilde{X}_t$ in the limit when the lattice spacing h tends to zero. To find this approximating process $\tilde{X}_t^h$, we introduce finite difference Markov generators defined on discrete lattices which approximate the differential generators in the continuum. The case of Levy processes is special and the mathematics is simpler because these processes are infinitely divisible, a property thanks to which the appropriate form of harmonic analysis is that based on ordinary Fourier transforms, thus simplifying the analysis.

The second class of examples we discuss involves non-divisible jump processes for which there is no translation invariance and as a consequence forms of harmonic analysis alternative to Fourier transforms are appropriate. We consider and compare four frameworks based on Laguerre, Meixner, Jacobi and Hahn polynomials, respectively. Non-divisible jump models are obtained by extending the construction of hypergeometric Brownian motions in [8], complementing the analysis with Bochner subordination and a lattice discretization. The fact that the construction extends is not obvious as the original proof in [8] does not extend to lattice models and needs to be modified. In section **??** we present a new argument which has the additional advantage of being extendible to lattice models.

Finally, we extend the framework to non-parametric models for which Markov generators can be diagonalised by means of numerical linear algebra. Our conclusion is that this latter class of non-parametric models is the most flexible and perhaps the most suitable to finance implementations. The paper concludes with a discussion of numerical examples.

2. HYPERGEOMETRIC BROWNIAN MOTIONS AND PROCESSES ON LATTICES

Hypergeometric Brownian motions were introduce in [8]. In this section, we give a new construction which has the advatnage of extending to the case of discrete state Markov processes.

We consider diffusion process X_t on some domain $D \subset \mathbb{R}$ defined by a Markov generator of the form

$$\mathcal{L}_X = \mathcal{L}^{\mathrm{P}} = m(x)\frac{d}{dx} + \frac{\sigma(x)^2}{2}\frac{d^2}{dx^2} \tag{2.1}$$

where $m(x)$ and $\sigma(x)$ are given smooth functions. The generator $\mathcal{L}_X$ is defined on the dense subspace of $L^1(D)$ spanned by the functions $f(x)$ satisfying linear boundary conditions of the form $af(x)+bf'(x) = 0$ at the boundary of he domain D. Under suitable assumptions on the boundary conditions, which depend on the asymptotic properties of $\sigma(x)$ and $\mu(x)$, (see Feller [**?**], the operator $\mathcal{L}_X$ generates a probability semigroup P_t

$$\mathcal{P}_t = e^{t\mathcal{L}}.$$

having the form of integral operators

$$\mathcal{P}_t f(x_0) = \int_D p_t(x_0, x_1) f(x_1) dx_1 \tag{2.2}$$

where $p_t(x_0, x_1)$ is the transitional probability density function. This semigroup satisfies the *Chapman-Kolmogorov equation*

$$\mathcal{P}_t \mathcal{P}_s = \mathcal{P}_{t+s}, \quad t, s \geq 0 \tag{2.3}$$

The probability density function satisfies the following equation *backward Kolmogorov equation*:

$$\frac{d}{dt} p_t(x_0, x_1) = \mathcal{L} p_t(x_0, x_1),$$

where the operator $\mathcal{L}$ acts on the first variable, x_0, and the initial time condition is $p_0(x_0, x_1) = \delta(x_1 - x_0)$. The transition probability measure for the process X_t is defined as follows:

$$\mathrm{P}(X_t \in A | X_0 = x_0) = \int\limits_A p_t(x_0, x_1) dx_1$$

Processes for which the probability density function is either known in analytically closed form or rapidly computable by means of numerical linear algebra or other means, play a central role in applications. Due to the fundamental theorem of Finance (see [9]), martingales (or, more generally, driftless processes) are the ones of most direct relevance to pricing theory. The next theorem allows one to construct driftless process Y_t out of general stochastic process X_t. This construction has the important property of preserving the analytical tractability of the model: the PDF of the process Y_t is given by a simple expression in terms of the PDF of the process X_t. We will first state this theorem for diffusions and then generalize it to include arbitrary stochastic processes and processes on the lattice.

Before we state the main theorem we need the following lemma:

Lemma 2.1. *For all $\rho \geq 0$, the equation*

$$\mathcal{L} f = \rho f. \tag{2.4}$$

has a strictly increasing solution $\varphi_\rho^+(x)$ and a strictly decreasing $\varphi_\rho^-(x)$ solutions. These solutions are convex, finite in the interior of the domain D and are related to the Laplace transform of the first hitting time as follows:

$$E_x\left[e^{-\lambda \tau_z}\right] = \begin{cases} \frac{\varphi_\rho^+(x)}{\varphi_\rho^+(z)}, & x \leq z, \\ \frac{\varphi_\rho^-(x)}{\varphi_\rho^-(z)}, & x \geq z. \end{cases} \tag{2.5}$$

Here $\tau_z = \inf\{t : X_t = z\}$ is the first hitting time of z.

This lemma can be proved by means of the Feynman-Kac formula (see [**?**]).

The main theorem of this section is stated as follows:

Theorem 2.2. *Let X_t be a diffusion process under the measure P taking values on the domain D and admitting a Markov generator $\mathcal{L} = \mathcal{L}^{\mathrm{P}}$ of the form in (2.1). Select a $\rho \geq 0$ and let $\varphi_\rho^+(x)$, $\varphi_\rho^-(x)$ be two positive, linearly independent solutions to the differential equation*

$$\mathcal{L} f = \rho f \tag{2.6}$$

on the domain D, $\varphi_\rho^+(x)$ increasing and $\varphi_\rho^-(x)$ decreasing. Take two constants $c_i \geq 0, \quad i = 1, 2$ and define a function

$$g(x) = c_1 \varphi_\rho^+(x) + c_2 \varphi_\rho^-(x) \tag{2.7}$$

Then we have the following:

(i) *there exists a measure* Q *absolutely continuous with respect to* P *defined by:*

$$\frac{d\mathrm{Q}_t}{d\mathrm{P}_t} = e^{-\rho t}\frac{g(X_t)}{g(X_0)}. \tag{2.8}$$

(ii) *The PDF of the process* X_t *under the measure* Q *is given by*

$$p_t^{\mathrm{Q}}(x_0, x_1) = e^{-\rho t}\frac{g(x_1)}{g(x_0)} p_t^{\mathrm{P}}(x_0, x_1)$$

(iii) *Markov generator* $\mathcal{L}^{\mathrm{Q}}$ *of the process* X_t *under the new measure* Q *is given by the following formula:*

$$\mathcal{L}^{\mathrm{Q}} = \frac{1}{g}\mathcal{L}^{\mathrm{P}} g - \rho.$$

(iv) *The function*

$$Y(x) = \frac{c_3\varphi_\rho^+(x) + c_4\varphi_\rho^-(x)}{g(x)} \tag{2.9}$$

is invertible for all choices of the constants c_3, c_4 *such that* $c_1c_4 - c_2c_3 \neq 0$ *and the process* $Y_t = Y(X_t)$ *is driftless under the measure* Q*. The process* Y_t *satisfies the stochastic differential equation*

$$dY_t = \eta(Y_t)dW_t^{\mathrm{Q}}, \tag{2.10}$$

where the volatility function $\eta(y)$ *is given by*

$$\eta(Y(x)) = \sigma(x)Y'(x) = \sigma(x)\frac{C}{g^2(x)}\exp\left(-2\int^x \frac{m(s)}{\sigma(s)^2}ds\right). \tag{2.11}$$

(v) *The probability distribution function for the process* Y_t *is given by*

$$p_t^Y(Y(x_0), Y(x_1)) = e^{-\rho t}\frac{g(x_1)}{g(x_0)} p_t^{\mathrm{P}}(x_0, x_1)Y'(x_1) \tag{2.12}$$

Proof. Since the function $g(x)$ satisfies $\mathcal{L}g = \rho g$, the process $M_t = e^{-\rho t}g(X_t)/g(X_0)$ is a positive local martingale, thus it is a supermartingale and equation (2.8) correctly specifies an absolutely continuous measure Q, thus proving (i).

To prove (ii), notice that for any bounded function f

$$E_0^{\mathrm{Q}}[f(X_t)] = E_0^{\mathrm{P}}\left[\frac{M_t}{M_0}f(X_t)\right] = \int_D e^{-\rho t}\frac{g(x_1)}{g(x_0)}p_t^{\mathrm{P}}(x_0, x_1)f(x_1)dx_1.$$

(iii) follows easily from (ii) and the definition of the Markov generator:

$$\mathcal{L}^{\mathrm{Q}} f(x_0) = \lim_{t\to 0^+}\frac{E_0^{\mathrm{Q}}[f(X_t)] - f(x_0)}{t}$$

The derivative $Y'(x)$ can be expressed through the Wronskian of the two independent solutions φ_ρ^+ and φ_ρ^-, namely

$$Y'(x) = (c_1c_4 - c_2c_3)\frac{W_{\varphi_\rho^+,\varphi_\rho^-}(x)}{g^2(x)} \tag{2.13}$$

where

$$W_{\varphi_\rho^+,\varphi_\rho^-}(x) = \det\begin{vmatrix}\varphi_\rho^+(x) & \varphi_\rho^-(x)\\ \varphi_\rho^+(x)' & \varphi_\rho^-(x)'\end{vmatrix} = C\exp\left(-2\int^x\frac{m(s)}{\sigma(s)^2}ds\right) \tag{2.14}$$

Hence, $Y'(x)$ is never zero and $Y(x)$ is invertible. This proves (iv) and shows that the volatility term in equation (2.11) is correct. To prove that the process $Y(X_t)$ is driftless we need to show that $\mathcal{L}^{\mathrm{Q}}Y(x) = 0$, which is easily verified:

$$\mathcal{L}^{\mathrm{Q}}Y(x) = \frac{1}{g(x)}\mathcal{L}^{\mathrm{P}}g(x)Y(x) - \rho Y(x) = \frac{\mathcal{L}^{\mathrm{P}}(c_3\varphi_\rho^+(x) + c_4\varphi_\rho^-(x))}{g(x)} - \rho Y(x) = \rho Y(x) - \rho Y(x) = 0,$$

since function $c_3\varphi_\rho^+(x) + c_4\varphi_\rho^-(x)$ is also a solution of $\mathcal{L}^{\mathrm{P}}f = \rho f$. □

2.1. **Processes on the Lattice .** In this section, we discuss lattice approximations of the processes discussed above.

If $D \subset \mathbb{R}$ is a domain and $h > 0$, let us introduce the following notation:

$$\Lambda(h, D) = (h\mathbb{Z}) \cap D \tag{2.15}$$

for its lattice discretization. One strategy to define discretized versions of Markov generators is to replace first and second space derivatives with a finite difference lattice operator as in the definition below:

Definition 2.3. If $h > 0$, let ∇^h_+ be the forward finite difference operator

$$\nabla^h_+ f(x) = \frac{f(x+h) - f(x)}{h} \tag{2.16}$$

and let Δ^h be the discretized Laplace operator such that

$$\Delta^h f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \tag{2.17}$$

The finite difference equivalents of the diffusion generator $\mathcal{L}$ given by (2.1) are Markov generators of discrete state Markov processes of the form

$$\mathcal{L}^h = m^h(x)\nabla^h_+ + \frac{\sigma^h(x)^2}{2}\Delta^h, \tag{2.18}$$

where $m^h(x)$ and $\sigma^h(x)$ are functions defined on the lattice $\Lambda(h, D)$ and converging to $m(x)$ and $\sigma(x)$, respectively, in the limit as $h \to 0$.

Remark 2.4. Discretizing the Markov generator is equivalent to discretizing the spatial variable in the partial differential equation for X_t.

The transition probabilities for the process X^h_t are defined as

$$p^h_t(x_0, x_1) = \mathrm{P}(X_t = x_1 | X_0 = x_0).$$

This function satisfies the discrete analogue of the backward Kolmogorov equation

$$\frac{d}{dt} p^h_t(x_0, x_1) = \mathcal{L}^h p^h_t(x_0, x_1),$$

where the operator $\mathcal{L}^h$ acts in the first variable x_0 and the initial time condition is $p^h_0(x_0, x_1) = \delta_{x_0,x_1}$. As $h \to 0$ we have (under some technical conditions)

$$\frac{1}{h} p^h_t(x_0, x_1) \to p_t(x_0, x_1).$$

The following analogue of lemma (2.1) holds:

Lemma 2.5. *Let $A \subset \Lambda(h, D)$ and A is nonempty. Define*

$$\tau_A = \inf\{t \geq 0 : \ X_t \in A\} \tag{2.19}$$

be the first passage time to A and let

$$\varphi_{A,\rho}(x) = E_x\left[e^{-\rho\tau_A}\right]. \tag{2.20}$$

Then $\varphi(x) = \varphi_{A,\rho}(x)$ satisfies the difference equation

$$\begin{cases} \mathcal{L}^h \varphi(x) = \rho\varphi(x), & \textit{if } x \notin A, \\ \varphi(x) = 1, & \textit{if } x \in A. \end{cases} \tag{2.21}$$

For the proof of this lemma see [**?**]. This result is important for constructing martingale processes on the lattice, since it guarantees that for every $\rho \geq 0$ the existence of two positive linearly independent solutions to equation

$$\mathcal{L}^h \varphi(x) = \rho\varphi(x).$$

These solutions are taken as a starting point in our construction.

The following theorem is the discrete analogue of theorem (2.2):

Theorem 2.6. *Let X^h_t be a birth and death process under the measure* P *taking values on the domain* $\Lambda(h, D)$ *and admitting a Markov generator* $\mathcal{L}^h$ *of the form in (2.18). Select a* $\rho \geq 0$ *and let* $\varphi_1(x)$, $\varphi_2(x)$ *be two positive, linearly independent solutions to the finite difference equation*

$$\mathcal{L}^h \varphi = \rho \varphi \tag{2.22}$$

defined on $\Lambda(h, D)$. *Given two constants* $c_i \geq 0, \;\; i = 1, 2$, *let's define the function*

$$g(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x). \tag{2.23}$$

The following holds:

(i) *there exists a measure* Q *absolutely continuous with respect to* P *defined by:*

$$\frac{d\mathrm{Q}_t}{d\mathrm{P}_t} = e^{-\rho t} \frac{g(X^h_t)}{g(X^h_0)}. \tag{2.24}$$

(ii) *The transitional probabilities of the process* X^h_t *under the measure* Q *are given by*

$$p^{h,\mathrm{Q}}_t(x_0, x_1) = e^{-\rho t} \frac{g(x_1)}{g(x_0)} p^{h,\mathrm{P}}_t(x_0, x_1).$$

(iii) *The Markov generator* $\mathcal{L}^{h,\mathrm{Q}}$ *of the process* X^h_t *under the new measure* Q *is given by the following formula:*

$$\mathcal{L}^{h,\mathrm{Q}} = \frac{1}{g} \mathcal{L}^{h,\mathrm{P}} g - \rho.$$

(iv) *The function*

$$Y^h(x) = \frac{c_3 \varphi_1(x) + c_4 \varphi_2(x)}{g(x)} \tag{2.25}$$

is invertible if $\frac{2hm^h(x)}{\sigma^h(x)^2} \leq 1$ *for all* $x \in \Lambda(h, D)$. *The process* $Y_t = Y(X_t)$ *is driftless under the measure* Q.

Proof. The arguments given above to prove (i), (ii),(iii) and (iv) in the continuous case apply and extend also to the discrete case. The only statement that requires additional work is the one about invertibility of $Y(x)$. The proof is essentially the same, it uses the discrete Wronskian and the condition $\frac{2hm^h(x)}{\sigma^h(x)^2} \leq 1$ ensures that the Wronskian is not zero.

ADD DETAILS FROM THE THESIS OR REFERENCE IT

□

3. BOCHNER SUBORDINATORS

In this section, we review some basic concepts regarding stochastic subordinators and jump processes.

Definition 3.1. Let (Ω, P) denote a probability space endowed with a right continuous and complete filtration $\mathcal{F}_t$. A *stochastic time change process* T_t is defined as a right-continuous non-decreasing process started from 0 and with values in $[0, \infty)$.

We shall refer to t as to the *calendar time* and call $s = T_t$ the *financial time* coordinate. The relative pace at which financial time proceeds compared to calendar time as modeled by the subordinator T_t, varies stochastically. A distinguished role is played by a special class of time changed processes called Bochner subordinators.

Definition 3.2. Let T_t be a stochastic time change process. The process T_t is called a *subordinator* if it has independent and homogeneous increments, i.e. $T_{t+s} - T_t$ is independent of $\mathcal{F}_t$ and has the same distribution as T_s.

One can show (see [10]) that T_t is a Bochner subordinator if and only if

$$E\left[e^{-\lambda T_t}\right] = \int_0^\infty e^{-\lambda s} \rho_t(ds) = e^{-t\phi(\lambda)} \tag{3.1}$$

for some function $\phi(\lambda)$, called *Bernstein function*. As an example, recall the definition of the Gamma process, which is widely used as subordinator (for example in the variance-gamma model). In this case, the PDF of T_t is

$$\rho_t(ds) = \frac{(\frac{\mu}{\nu})^{\frac{\mu^2 t}{\nu}}}{\Gamma(\frac{\mu^2 t}{\nu})} s^{\frac{\mu^2 t}{\nu}-1} e^{-\frac{\mu}{\nu}s} ds \tag{3.2}$$

where μ is the mean rate and ν is the variance rate. The corresponding Bernstein function is given by

$$\phi(\lambda) = \frac{\mu^2}{\nu} \log\left(1 + \lambda\frac{\nu}{\mu}\right). \tag{3.3}$$

Remark 3.3. Subordinators could be equivalently defined as increasing Levy processes. Note that the Levy process is increasing if and only if it has positive drift, only positive jumps and no diffusion component.

Definition 3.4. Let X_t be a stochastic process and T_t be the time change process (subordinator). The time-changed (subordinated process) $\tilde{X}_t$ is defined as

$$\tilde{X}_t = X_{T_t}.$$

One can also show (see [11]) that if $\mathcal{L}$ is the Markov generator of a given process X_t and $\phi(\lambda)$ is the Bernstein function for the subordinator T_t, then functional calculus can be used to express the generator of the time changed process $\tilde{X}_t = X_{T_t}$ as

$$\tilde{\mathcal{L}} = -\phi(-\mathcal{L}). \tag{3.4}$$

In fact, the probability semigroup for X_t is given by $\exp(t\mathcal{L})$ and thus the probability semigroup for X_{T_t} can be computed as $E[\exp(T_t\mathcal{L})]$. Applying equation 3.1, this becomes

$$E\left[e^{T_t\mathcal{L}}\right] = e^{-t\phi(-\mathcal{L})},$$

thus the new generator is $\tilde{\mathcal{L}} = -\phi(-\mathcal{L})$.

Recall that if the operator $\mathcal{L}$ is bounded and $\phi(z) = \sum_{n=0}^{\infty} \phi_n z^n$ is an entire analytic function, one can define the operator $\phi(-\mathcal{L})$ by means of a series expansion convergent in operator norm, i.e. by setting

$$-\phi(-\mathcal{L}) = \sum_{n=0}^{\infty} (-1)^{n+1} \phi_n \mathcal{L}^n \tag{3.5}$$

Functional calculus extends also to the case where the Markov generator $\mathcal{L}$ has discrete spectrum. Suppose that $\psi_n(x)$ where $n = 0, 1, ...$ is a family of eigenfunctions of $\mathcal{L}$ with eigenvalues λ_n, i.e. suppose that

$$\mathcal{L}\psi_n(x) = \lambda_n \psi_n(x). \tag{3.6}$$

Suppose also that $\psi_n(x)$ is a complete orthonormal basis of $L^2_\nu(\mathbb{R})$ for some measure $\nu(dx)$, in the sense that

$$\int \psi_n(x)\psi_m(x)\nu(dx) = \delta_{nm}. \tag{3.7}$$

Remark 3.5. If one can find an orthonormal basis of eigenvectors for the Markov generator $\mathcal{L}$, then one is able to compute explicitly the probability density for the process X_t as

$$p_t(x_0, x_1) = \sum_{n=0}^{\infty} e^{t\lambda_n} \psi_n(x_0)\psi_n(x_1)\nu(dx_1). \tag{3.8}$$

This expansion can be justified by noting that the function $p_t(x_0, x_1)$ is the integral kernel of the probability semigroup $\mathcal{P}_t = \exp(t\mathcal{L})$.

The operator $\tilde{\mathcal{L}} = -\phi(-\mathcal{L})$ can be defined as the operator having the same eigenfunctions as $\mathcal{L}$ and satisfying the eigenvalue equation

$$\tilde{\mathcal{L}}\psi_n(x) = -\phi(-\lambda_n)\psi_n(x). \tag{3.9}$$

Hence, the probability distribution function for X_{T_t}, which is the integral kernel of the semigroup $\tilde{\mathcal{P}}_t = \exp(t\tilde{\mathcal{L}})$, can be computed as follows:

$$\tilde{p}_t(x_0, x_1) = \sum_{n=0}^{\infty} e^{-t\phi(-\lambda_n)} \psi_n(x_0)\psi_n(x_1)\nu(dx_1). \tag{3.10}$$

Notice that this equation is very similar to equation (3.8), thus this method of time change is very efficient, provided we know how to diagonalize the Markov generator $\mathcal{L}$.

Functional calculus also extends to cases where the Markov generator $\mathcal{L}$ is unbounded, to functions $\phi(\lambda)$ which are not analytic in λ and to cases where the spectrum of $\mathcal{L}$ is not purely discrete. A Markov generator $\mathcal{L}$ satisfying the reversibility condition

$$\int f(\mathcal{L}g)\nu(dx) = \int (\mathcal{L}f)g\nu(dx) \tag{3.11}$$

for some measure $\nu(dx)$ is symmetric in the Hilbert space $L^2_\nu(\mathbb{R})$. Such an operator with reflecting boundary conditions on the end-points of D is self-adjoint, see [12], and admits a spectral resolution of the form

$$\mathcal{L} = \int \lambda \mu(dP_\lambda) \tag{3.12}$$

where $\mu(dP_\lambda)$ is a projection valued measure. We refer again to [12] for a definition of projection valued measure. As an example, we shall quote here the case of the Brownian motion with the generator $\mathcal{L} = \frac{d^2}{dx^2}$. In this case, the kernel of $\mathcal{L}$ is a generalized function given by the second derivative of a delta function and can be represented as follows:

$$\mathcal{L}(x,y) = \delta''(x-y) = -\int_{-\infty}^{\infty} \frac{dk}{2\pi}\ k^2 e^{ik(x-y)}. \tag{3.13}$$

This equation is in the form that the general spectral decomposition formula in (3.13) takes in this particular case, showing that $\mu(dP_\lambda)$ in this case is the measure supported on $\lambda \in [0\infty]$ whose kernel is given by

$$\mu(dP_\lambda)(x,y) = e^{i\sqrt{\lambda}(x-y)} \frac{d\lambda}{4\pi\sqrt{\lambda}}. \tag{3.14}$$

If $\phi(\lambda)$ is Bernstein function, then the operator $\phi(\mathcal{L})$ can be defined by means of the spectral resolution formula

$$\tilde{\mathcal{L}} = -\int \phi(-\lambda)\mu(dP_\lambda) \tag{3.15}$$

as long as the integral appearing here converges to a self-adjoint operator.

4. LEVY PROCESSES AND SUBORDINATED BROWNIAN MOTION

Levy processes provide the simplest example for the application of the methods in this paper. The reason is that a subordinated Levy process is also a Levy process. The widely used example is variance-gamma model, introduced by Madan and Seneta [4] and later extended in [3]. In this model the Brownian motion with drift subordinated by a Gamma process becomes the Levy process, which is the difference of two independent Gamma processes.

Levy processes were considered in the finance literature by Clark [13] and then studied by Geman et al. [14], Gorieroux et al. [15]. If F_t is a martingale process we wish to describe with a Levy model, for instance a forward price under the corresponding forward measure, the process F_t has the form

$$F_t = F_0 \exp\left(\tilde{X}_t\right) \tag{4.1}$$

where

$$\tilde{X}_t \equiv \omega t + X_{T_t}, \quad X_t = \theta t + \sigma W_t. \tag{4.2}$$

The martingale condition results in a constraint among parameters. For instance, in the case of the variance-gamma model where the Bernstein function is given by (3.3), the process F_t is a martingale if ω is chosen as follows:

$$\omega = \frac{1}{\nu} \log\left(1 - \theta\nu - \frac{\sigma^2\nu}{2}\right) \tag{4.3}$$

Theorem 4.1. *The probability distribution function for the Levy process $\tilde{X}_t$ defined in equation (4.2) where the subordinator is characterized by the Bernstein function $\phi(\lambda)$ can be expressed as follows:*

$$\tilde{p}_t(x_0, x_1) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{t\tilde{\epsilon}(k;\sigma,\theta,\omega)+ik(x_0-x_1)} \tag{4.4}$$

where

$$\tilde{\epsilon}(k;\sigma,\theta,\omega) = i\omega k - \phi\left(\frac{\sigma^2k^2}{2} - ik\theta\right). \tag{4.5}$$

The following family of probability functions on $\Lambda(h,\mathbb{R})$, $h > 0$, is a pointwise approximation:

$$\tilde{p}_t^h(x_0, x_1) = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \frac{dk}{2\pi} e^{t\tilde{\epsilon}^h(k;\sigma,\theta,\omega)+ik(x_0-x_1)}. \tag{4.6}$$

where

$$\tilde{\epsilon}^h(k;\sigma,\theta,\omega) = \frac{i\omega\sin(kh)}{h} - \phi\left(-\sigma^2(\cos(kah) - 1) - \frac{i\theta\sin(kh)}{h}\right). \tag{4.7}$$

Proof. The Markov generator corresponding to the equation in (4.2) is the operator

$$\tilde{\mathcal{L}} = \omega\frac{d}{dx} - \phi(-\mathcal{L}) \quad \text{where} \quad \mathcal{L} = \frac{\sigma^2}{2}\frac{d^2}{dx^2} + \theta\frac{d}{dx}. \tag{4.8}$$

The eigenfunctions of $\mathcal{L}$ in this case are given by e^{ikx}, $k \in \mathbb{R}$ with eigenvalues $\epsilon(k) = -\frac{\sigma^2k^2}{2} + ik\theta$. These eigenfunctions are not normalizable since $x \in \mathbb{R}$. The generalized eigenvalue equation for $\tilde{\mathcal{L}}$ is

$$\tilde{\mathcal{L}}e^{ikx} = \tilde{\epsilon}(k)e^{ikx} \tag{4.9}$$

where $\tilde{\epsilon}(k;\sigma,\theta,\omega)$ is given by equation (4.5). Hence, the probability distribution function of the subordinated process is

$$\tilde{p}_t(x_0, x_1) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{t\tilde{\epsilon}(k;\sigma,\theta,\omega)+ik(x_0-x_1)} \tag{4.10}$$

The Markov generator $\tilde{\mathcal{L}}^h$ of the process $\tilde{X}_t^h$ on the lattice, which approximates $\tilde{X}_t$, can be defined as follows:

$$\tilde{\mathcal{L}}^h = \omega\nabla^h - \phi(-\mathcal{L}^h) \quad \text{where} \quad \mathcal{L}^h = \frac{\sigma^2}{2}\Delta^h + \theta\nabla^h. \tag{4.11}$$

The generalized eigenfunctions of the discretized Markov generators $\mathcal{L}^h$ and $\tilde{\mathcal{L}}^h$ are still given by the functions e^{ikx}, except that in case $h > 0$ the wave-vector k is restricted to the range $(-\frac{\pi}{h}, \frac{\pi}{h})$. The generalized eigenvalues of the operator $\mathcal{L}^h$ are given by

$$\epsilon^h(k;\sigma,\theta) = \sigma^2(\cos(kah) - 1) + \frac{i\theta\sin(kh)}{h}. \tag{4.12}$$

The eigenvalues of the operator $\phi(-\mathcal{L}^h)$ are thus $\phi(-\epsilon^h(k;\sigma,\theta))$. Hence, the generalized eigenvalues of the operator $\tilde{\mathcal{L}}^h = \omega\frac{d}{dx} - \phi(-\mathcal{L}^h)$ are given by the expression in equation (4.7). We thus obtain the representation (4.6) for the PDF of the process X_t^h. Thanks to the theorem of dominated convergence and using the explicit expressions for the kernels in (4.6) and (4.4), we then conclude that we have pointwise convergence, i.e.

$$\lim_{h\to 0} \tilde{p}_t^h(x_0, x_1) = \tilde{p}_t(x_0, x_1). \tag{4.13}$$

$\square$

5. SUBORDINATED HYPERGEOMETRIC BROWNIAN MOTIONS AND PROCESSES ON LATTICES

In this section we provide a general algorithm to evaluate probability transition kernels for a class of driftless diffusions and their counterparts on the lattice. As we discussed, the main ingredient is the orthonormal basis of eigenvectors for the Markov generator $\mathcal{L}$. Thus we start with a generator $\mathcal{L}$ with known eigenvectors $\psi_n(x)$ and eigenvalues λ_n. In particular, we are interested in the processes for which $\psi_n(x)$ is a set of polynomials, orthogonal with respect to some measure $\nu(dx)$. It turns out (see ADD REFERENCE) that the only diffusion processes, associated with a family of orthogonal polynomials are Ornstein-Uhlenbeck, CIR and Jacobi diffusions. For details on these processes and their discrete counterparts (Charlier, Meixner and Hahn birth and death processes) see Appendix ...

First we need to understand what is the effect of transformation discussed in section 2 on eigenvectors of Markov generator. We know that after change of measure given by equation 2.8 the generator transforms into

$$\mathcal{L}^{\mathrm{Q}} = \frac{1}{g}\mathcal{L}^{\mathrm{P}} g - \rho,$$

thus the eigenfunctions $\psi_n(x)$ and eigenvalues λ_n transform as follows

$$\psi_n(x) \mapsto \frac{\psi_n(x)}{g(x)}, \quad \lambda_n \mapsto \lambda_n - \rho.$$

The orthogonality measure $\nu(dx)$ is transforms into $g^2(x)\nu(dx)$.

Change of variables do not change eigenvalues and affect eigenvectors and orthogonality measures as follows:

$$\psi_n(x) \mapsto \psi_n(X(y)), \quad \nu(dx) \mapsto \nu(X(dy)).$$

Notice that there is no derivative in the formula for the measure in the discrete case.

Thus we obtain the expression for the transition kernel of the process Y_t as follows:

$$p_t^Y(Y(x_0), Y(x_1)) = \sum_{n=0}^{\infty} e^{t(\lambda_n - \rho)} \frac{\psi_n(x_0)}{g(x_0)} \frac{\psi_n(x_1)}{g(x_1)} g^2(x_1)\nu(dx_1), \tag{5.1}$$

which is of course equivalent to what given in equation 2.12.

At last we can subordinate the process Y_t to obtain $\tilde{Y}_t = Y_{T_t}$. Since the eigenvectors and eigenvalues of the Markov generator for the process Y_t are known, the kernel for $\tilde{Y}_t$ is given by

$$p_t^{\tilde{Y}}(Y(x_0), Y(x_1)) = \frac{g(x_1)}{g(x_0)} \sum_{n=0}^{\infty} e^{-t\phi(-\lambda_n + \rho)} \psi_n(x_0)\psi_n(x_1)\nu(dx_1). \tag{5.2}$$

The algorithm for constructing analytically solvable time-changed driftless process and their lattice approximations can thus be described as follows:

- Let X_t be the process, associated with a family of orthogonal polynomials.
- Fix $\rho \geq 0$ and find two positive, linearly independent solutions $\varphi_\rho^+, \varphi_\rho^-$ to equation

  $$\mathcal{L}\varphi = \rho\varphi.$$

  In the diffusion case these function are given in terms of hypergeometric functions (see ?????). For processes on the lattice the above equation is a three point finite difference equation and can be solved by simple recursion.
- Choose nonnegative constants c_1 and c_2 defining functions $g = c_1\varphi_\rho^+ + c_2\varphi_\rho^-$ and constants c_3, c_4 defining function $Y(x) = (c_3\varphi_\rho^+ + c_4\varphi_\rho^-)/g$.
- Choose the subordinator T_t, preferably in such a way so that Bernstein function $\phi(\lambda)$ can be easily computed. Important examples are provided by Gamma process and stable processes.
- Compute the transitional probabilities for process $\tilde{Y}_t = Y_{T_t}$ using formula 5.2.

Remark 5.1. The measure change defined by equation 2.8 in most cases leads to exit or killing boundary conditions, thus the transformed process Y_t is not conservative. This problem can be avoided to some extent by ensuring that Y_t starts away from the boundaries. Also note that ρ is the rate at which the process is killed at the boundary ($-\rho$ is the maximum eigenvalue of generator of Y_t), thus in order to keep this effect under control one should take ρ sufficiently small.

6. NON PARAMETRIC EXTENSIONS BY NUMERICAL DIAGONALIZATION

As we have seen in the previous sections, the ability to perform time changes and effectively compute the transitional probabilities hinges on our ability to diagonalize the Markov generator $\mathcal{L}$. In the previous sections we discussed processes having a given set of orthogonal polynomials as eigenvectors of the Markov generator $\mathcal{L}$. In the discrete case, these processes are useful since we know their continuous limit. In other situations though, knowing that the continuous limit exists is sufficient information and it is convenient to define the process non parametrically and to compute eigenfunctions using numerical linear algebra.

In a non-parametric approach, the first step is to define the lattice. We assume that the lattice Λ is finite and has N nodes, and that the position of the nodes on the real line is given by $x_i = F(i),\ i = 1 \dots N$ for some increasing function F.

We specify the process X_t on the lattice Λ using it's Markov generator, which in finite case is $N \times N$ matrix. As in the previous sections, it is useful to start with the discrete analog of a diffusion processes and a tridiagonal generator in the form

$$\mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ L_{21} & -L_{22} & L_{23} & 0 & \dots & 0 & 0 & 0 \\ 0 & L_{32} & -L_{33} & L_{34} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -L_{N-2N-2} & L_{N-2N-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & L_{N-1N-2} & -L_{N-1N-1} & L_{N-1N} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \tag{6.1}$$

Note that X_t specified by $\mathcal{L}$ has absorbing boundaries. This is relevant to the constructions below as we focus on martingale process based on X_t. The generator $\mathcal{L}$ also has to satisfy the following two conditions:

- $\sum_j L_{ij} = 0$, which ensures the conservation of probability
- $\sum_{j \neq i} L_{ij}(F(j) - F(i))^2 = F^2(i)\sigma^2(F(i))$, which specifies the "local volatility" of the process

Each row of $\mathcal{L}$ has three elements and two conditions given above, thus we have one degree of freedom, which is set by choosing the state-dependent "drift" of the process.

To subordinate the process X_t, as described in the previous sections, first we need to diagonalize the generator $\mathcal{L}$. Since $\mathcal{L}$ is a finite matrix, one could use the usual linear algebra algorithms provided N is not too large. Thus we arrive at the set of eigenvectors $\psi_i \in \mathbb{R}^N,\ i = 1 \dots N$ and eigenvalues λ_i, such that

$$\mathcal{L}\psi_i = \lambda_i \psi_i, \quad i = 1 \dots N.$$

If Ψ is the matrix which has ψ_i as columns, we find

$$\mathcal{L} = \Psi D \Psi^{-1}, \quad D = \text{diag}\{\lambda_i\}.$$

The generator for the subordinated process $\tilde{X}_t = X_{T_t}$ is computed as follows:

$$\tilde{\mathcal{L}} f = -\phi(-\mathcal{L}) f = \Psi \tilde{D} \Psi^{-1}, \quad \tilde{D} = \text{diag}\{-\phi(-\lambda_i)\}, \tag{6.2}$$

where $\phi(\lambda)$ is the Bernstein function of the subordinator.

Note that original process X_t is defined to have state-dependent drift, so that the martingale condition

$$\sum_j \tilde{L}_{ij} F(j) = 0$$

for the process $\tilde{X}_t$ is violated. One can restore the martingale condition by changing the off-diagonal coefficients of $\tilde{L}$, which amounts to introducing a drift similarly to what is done in the definition of the variance-gamma model. If for some i there is a nonzero drift in the process $\tilde{X}_t$ is positive:

$$\sum_j \tilde{L}_{ij} F(j) = d_i > 0,$$

then we add the positive amount $d_i/(F(i) - F(i-1))$ to the element $\tilde{L}_{ii-1}$ and subtract the same from $\tilde{L}_{ii-1}$. If d_i is negative, one adds $d_i/(F(i+1) - F(i))$ to the elements $\tilde{L}_{ii+1}$ and subtracct it from $\tilde{L}_{ii}$. This procedure of adding a drift term leads to a martingale process Y_t on the lattice Λ.

This idea is very similar and extends the one at the basis of the construction of the variance-gamma model: one adds an independent pure drift process Π_t to offset the drift in $\tilde{X}_t$. In the VG model $\Pi_t = wt$, since all the processes are space-homogeneous (Levy) processes. In this more general example Π_t is state dependent process; at state i it can jump only upwards ($d_i < 0$) or downward ($d_i > 0$) to the nearest state.

Having constructed the martingale process Y_t having generator $\mathcal{L}_Y$, we again diagonalize it to obtain

$$\mathcal{L}_Y = \hat{\Psi}\hat{D}\hat{\Psi}^{-1}, \hat{D} = \text{diag}\{\hat{\lambda}_i\}$$

and the matrix of transition probabilities $\mathcal{P}_t^Y$ is easily computed as follows:

$$\mathcal{P}_t^Y = \hat{\Psi} e^{t\hat{D}} \hat{\Psi}^{-1}, e^{t\hat{D}} = \text{diag}\{e^{t\hat{\lambda}_i}\} \tag{6.3}$$

Let's summarize the construction:

- Start with a process X_t on the lattice $\Lambda = \{F(i),\ i = 1 \dots N\}$ admitting only nearest neighbor transitions. Assume the boundaries are absorbing.
- Diagonalize the Markov generator of X_t
- Subordinate X_t to obtain $\tilde{X}_t = X_{T_t}$. If there is any drift in X_t it would create asymmetric jumps in $\tilde{X}_t$.
- Find the generator $\tilde{\mathcal{L}}$ for $\tilde{X}_t$ using formula 7.2
- Remove the drift in $\tilde{X}_t$ by changing the generator $\tilde{\mathcal{L}} \mapsto \mathcal{L}_Y$.
- Y_t is the desired martingale process. To perform computations, diagonalize $\mathcal{L}_Y$ and use formula 7.3.

Remark 6.1. In the third step of the above algorithm, one subordinates the process by an independent time change T_t. However, in the one could extend the construction to allow for time changes correlated with the process X_t. A possible construction is the following: write X_t as the sum of two processes

$$X_t = X_t^+ + X_t^-,$$

where X^+ (X^-) is an increasing (decreasing) process. Their generators can be easily found by breaking $\mathcal{L}$ into the sum of the upper-triangular and lower-triangular generators $\mathcal{L}^+$ and $\mathcal{L}^-$.

One can then subordinate each of X^+, X^- separately:

$$\tilde{X}_t^+ = X^+_{T_t^1}, \quad \tilde{X}_t^- = X^-_{T_t^2},$$

and then finally construct $\tilde{X}_t$ as

$$\tilde{X}_t = \tilde{X}_t^+ + \tilde{X}_t^-.$$

This construction is possible because the processes involved are of finite variation. It is also easy to implement efficiently as it involves just one extra diagonalization.

The algorithm described above is very flexible and useful

HERE WE PUT EUROPEAN PRICES, AND DISCUSS HOW COOL THIS METHOD IS

Another immediate application of this nonparametric method is pricing of barrier options. Assume we want to price a down-and-out call option struck at K with a barrier at $D = F(l) < K$. The price of the option in this case is

$$C_T = E[I\{Y_s > D,\ s \in [0,T]\}(Y_T - K)^+].$$

We can reformulate this problem by introducing another process Y_t^0, which is basically Y_t with absorption in the region $y \le D$. Then the price can be rewritten as

$$C_T = E[(Y_T^0 - K)^+],$$

which is now a usual European option on process Y_t^0. The generator $\mathcal{L}_{Y^0}$ for Y_t^0 can be obtained from $\mathcal{L}_Y$ by replacing all the rows with numbers $i \le l$ by zeros:

$$\begin{cases} L_{ij}^0 = L_{ij}^Y, & i > l \\ L_{ij}^0 = 0, & i \le l \end{cases}$$

Again, by diagonalizing the generator $\mathcal{L}_{Y^0}$ we can easily compute the transitional probabilities for process Y^0 and thus find prices of barrier option.

HERE WE PUT THE GRAPHS RELATED TO BARRIER OPTIONS

7. Concluding Remarks

In this article we introduce a new discretization framework for PIDE's arising from pricing equations based on jump processes. The scheme is based on an approximation for the underlying price process. We focus on a broad class of models which is analytically tractable and compute transition probabilities in analytically closed form in terms of finite expansions in orthogonal polynomials. The number of terms in the series is bounded by the number of lattice sites and this ensures rapid convergence. The main advantage of this discretization scheme is that results don't depend on the size of time steps one chooses to use while local no-arbitrage conditions are maintained.

References

[1] R. C. Merton. Option pricing when underlying stock returns are discontinuous. *J. of Financial Economics.*, 1976.
[2] E. Eberlein and U. Keller. Hyperbolic distributions in finance. *Bernoulli*, pages 281–299, 1995.
[3] D. Madan, P. Carr, and E. Chang. The variance gamma process and option pricing. *European Finance Review*, 2(1):79–105, 1998.
[4] D. Madan and E. Seneta. The variance gamma model for share market returns. *Journal of Business*, 1990.
[5] D. Madan and A. Hirsa. Pricing American Options under Variance Gamma. *Journal of Computational Finance*, Forthcoming in 2003.
[6] T. Petersdorff and Schwab C. Wavelet discretizations of parabolic integro-differential equations. *SIAM Journal of Numerical Analysis*, 2003.
[7] A.M. Matache, T. Petersdorff, and Schwab C. Fast deterministic pricing of options on levy driven assets. *ETHZ, preprint*, 2002.
[8] C. Albanese, G. Camplolieti, P. Carr, and A. Lipton. Black-Scholes goes hypergeometric. *Risk*, 14(12):99–103, 2001.
[9] F. Delbaen and W. Schachermeyer. A general version of the fundamental theorem of asset pricing. *Math. Ann.*, 1994.
[10] S. Bochner. *Harmonic analysis and the theory of probability*. University of California Press, Berkeley, 1955.
[11] R.S. Phillips. On the generation of semigroups of linear operators. *Pacific J. Math.*, pages 343–369, 1952.
[12] M. Reed and B. Simon. *Methods of Modern Mathematical Physics, Vol. 2*. Academic Press, Boston, 1980.
[13] Clark P. A subordinated stochastic process model with finite variance for speculative prices. *Econometrica 41*, 1973.
[14] H. Geman and T. Ane. Stochastic subordination. *Risk*, September, 1996.
[15] G. Gourieroux, C., G. Le Fol, and B. Meyer. Analysis of order queues. *Working paper, CREST, Paris*, 1996.
[16] J.C. Cox, Ingersoll J.E., and S.A. Ross. A theory of term structure of interest rates. *Econometrica*, 1985.
[17] W. Gautschi. Orthpol – a package of routines for generating orthogonal polynomials and gauss-type quadrature rules. *ACM Trans. Math. Software*, pages 21–62, 1994.
[18] W.A. Al-Salam. *Characterization theorems for orthogonal polynomials. In : Orthogonal Polynomials : Theory and Practice (ed. P. Nevai)*. Kluwer Academic Publishers, Dordrecht, 1990.
[19] G.E. Andrews and R. Askey. *Classical orthogonal polynomials. In : Polynomes Orthogonaux et Applications (eds. C. Brezinski et al.). Lecture Notes in Mathematics 1171*. Springer-Verlag, New York, 1985.

Appendix A. Hypergeometric functions

In this section we review several facts about hypergeometric functions, such as Taylor series, differential equation and increasing/decreasing solutions to this equation. A good collection of facts and formulas can be found in [**?**] and [**?**].

Hypergeometric functions are defined through Taylor series expansion which generalize the geometric series

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)...(\alpha+n-1)}{n!} z^n = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n. \tag{A.1}$$

Generalizing this expansion, we introduce the functions

$$ {}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ...\beta_q; z), \quad p \le q+1, \beta_j \in \mathbb{C} \setminus -\mathbb{Z}_+ \tag{A.2} $$

which, for $|z| < 1$, is the sum of the series

$$ {}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ...\beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n....(\alpha_p)_n}{n!(\beta_1)_n..(\beta_q)_n} z^n, \tag{A.3} $$

where the *Pochhammer symbol* $(\alpha)_n$ is defined by

$$ (\alpha)_n = \alpha(\alpha+1)\dots(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \quad (\alpha)_0 = 1. $$

APPENDIX A.1. **Hypergeometric function.** The hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$ has Taylor expansion at 0

$$ {}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!}. \tag{A.4} $$

This function is a solution to the *hypergeometric differential equation*

$$ z(1-z)F''(z) + (\gamma - (1+\alpha+\beta)z)F'(z) - \alpha\beta F(z) = 0. \tag{A.5} $$

Two linearly independent solution in the neighborhood of $z = 0$ are given by:

$$ w_1 = {}_2F_1(\alpha, \beta; \gamma; z), \tag{A.6} $$

$$ w_2 = z^{1-\gamma} {}_2F_1(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z), \tag{A.7} $$

and in the neighborhood of $z = 1$

$$ w_1 = {}_2F_1(\alpha, \beta; \alpha + \beta + 1 - \gamma; 1 - z), \tag{A.8} $$

$$ w_2 = (1-z)^{\gamma - \alpha - \beta} {}_2F_1(\gamma - \beta, \gamma - \alpha, \gamma - \alpha - \beta + 1, 1 - z). \tag{A.9} $$

Increasing and decreasing solutions of the hypergeometric equation, which in the case $\alpha > 0, \ \beta > 0, \ \gamma > 0$ and $\gamma < \alpha + \beta + 1$ are given by:

$$ \varphi^+(x) = {}_2F_1(\alpha, \beta; \gamma; z), \tag{A.10} $$

$$ \varphi^-(x) = {}_2F_1(\alpha, \beta; \alpha + \beta + 1 - \gamma; 1 - z). \tag{A.11} $$

APPENDIX A.2. **Confluent hypergeometric function.** The confluent hypergeometric function ${}_1F_1(a, b, z)$ (also denoted as $M(a, b, z)$) has Taylor expansion

$$ M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}. \tag{A.12} $$

Function $M(a, b, z)$ is a solution to the *Kummer differential equation*

$$ zF''(z) + (b - z)F'(z) - aF(z) = 0 \tag{A.13} $$

Two linearly independent solutions to the Kummer differential equation are given by:

$$ w_1 = M(a; b; z), \quad w_2 = z^{1-b} M(1 + a - b; 2 - b; z). \tag{A.14} $$

The increasing and decreasing solutions are given by:

$$ \varphi^+(x) = M(a; b; z), \tag{A.15} $$

$$ \varphi^-(x) = U(a, b, z) = \frac{\pi}{\sin(\pi b)} \left(\frac{M(a, b, z)}{\Gamma(1 + a - b)\Gamma(b)} - z^{1-b} \frac{M(1 + a - b, 2 - b, z)}{\Gamma(a)\Gamma(2 - b)} \right). \tag{A.16} $$

APPENDIX B. CIR, MEIXNER, JACOBI AND HAHN PROCESSES

APPENDIX B.1. **CIR process.**

- Generator

$$\mathcal{L} = (a - bx)\frac{d}{dx} + \frac{1}{2}\sigma^2 x \frac{d^2}{dx^2}. \tag{B.1}$$

- Domain $D = [0, +\infty)$
- Speed measure and scale function:

$$\nu(x) = \frac{\theta^{\alpha+1}}{\Gamma(\alpha+1)} x^\alpha e^{-\theta x}, \quad s'(x) = x^{-\alpha-1} e^{\theta x}, \tag{B.2}$$

where $\alpha = \frac{2a}{\sigma^2} - 1$ and $\theta = \frac{2b}{\sigma^2}$.
- Boundary conditions: $D^2 = +\infty$ is a natural boundary for all choices of parameters and

$$D^1 = \begin{cases} \text{exit, if } \alpha \le -1 \\ \text{regular, if } -1 < \alpha < 0 \\ \text{entrance, if } 0 \le \alpha \end{cases} \tag{B.3}$$

- Probability function:

$$p_t^{(\mathrm{CIR})}(x_0, x_1) = c_t \left(\frac{x_1 e^{bt}}{x_0}\right)^{\frac{1}{2}\alpha} \exp\left[-c_t(x_0 e^{-bt} + x_1)\right] I_\alpha\left(2c_t\sqrt{x_0 x_1 e^{-bt}}\right), \tag{B.4}$$

where $c_t \equiv -2b/(\sigma^2(e^{-bt} - 1))$ and I_α is the modified Bessel function of the first kind (see [**?**]).
- Spectrum of the generator:

$$\lambda_n = -bn. \tag{B.5}$$

- Eigenfunctions of the generator:

$$\psi_n(x) = L_n^\alpha(\theta x), \tag{B.6}$$

where $L_n^\alpha(y)$ are Laguerre polynomials of order α with the three term recurrence relation:

$$(n+1)L_{n+1}^\alpha(y) - (2n + \alpha + 1 - y)L_n^\alpha(y) + (n + \alpha)L_{n-1}^\alpha(y) = 0. \tag{B.7}$$

- Orthogonality relation:

$$\int_D \psi_n(x)\psi_m(x)\nu(x)dx = \frac{(\alpha+1)_n}{n!}\delta_{nm}.$$

- Eigenfunction expansion of the probability function:

$$p_t^{(\mathrm{CIR})}(x_0, x_1) = \sum_{n=0}^{\infty} e^{-bnt} \frac{n!}{(\alpha+1)_n} L_n^\alpha(\theta x_0) L_n^\alpha(\theta x_1)\nu(x_1). \tag{B.8}$$

APPENDIX B.2. **Meixner process.**

- Generator

$$\mathcal{L} = (a - bx)\nabla_\delta^+ + \frac{1}{2}\sigma^2 x \Delta_\delta. \tag{B.9}$$

- Domain $D = \{0, \delta, 2\delta, \dots\}$
- Spectrum of the generator:

$$\lambda_n = -bn. \tag{B.10}$$

- Eigenfunctions of the generator:

$$\psi_n(x) = M_n(x/\delta, \beta, c), \tag{B.11}$$

where $c = 1 - \frac{2b\delta}{\sigma^2}$ and $\beta = \frac{2a}{c\sigma^2}$ and $M_n(y, \beta, c)$ are Meixner polynomials with the three term recurrence relation:

$$(c-1)yM_n(y,\beta,c) = c(n+\beta)M_{n+1}(y,\beta,c) - (n + (n+\beta)c)M_n(y,\beta,c) + nM_{n-1}(y,\beta,c).$$

- Orthogonality measure (invariant measure) is given by the *negative binomial distribution* or *Pascal distribution* $Pa(\beta, c)$:

$$\nu(\{x\}) = \frac{(\beta)_y c^y}{y!(1-c)^\beta} \tag{B.12}$$

and orthogonality relation

$$\sum_{y=0}^{\infty} M_n(y,\beta,c) M_m(y,\beta,c) nu(\{x\}) = \frac{c^{-n} n!}{(\beta)_n}\delta_{nm},$$

where $y = x/\delta$.

- Eigenfunction expansion of the probability function:

$$p_t^{\text{(Meixner)}}(x_0, x_1) = \sum_{n=0}^{\infty} e^{-bnt}\frac{c^n(\beta)_n}{n!} M_n(y_0,\beta,c) M_n(y_1,\beta,c)\nu(\{y1\}), \tag{B.13}$$

where $y_i = x_i/\delta \in \{0, 1, 2, \dots\}$.

APPENDIX B.3. **Jacobi process.**

- Generator

$$\mathcal{L} = (a - bx)\frac{d}{dx} + \frac{1}{2}\sigma^2 x(A - x)\frac{d^2}{dx^2}. \tag{B.14}$$

- Domain $D = [0, A]$
- Speed measure and scale function:

$$\nu(x) = \frac{x^\beta (A-x)^\alpha}{A^{\alpha+\beta+1} B(\alpha+1, \beta+1)}, \quad s'(x) = x^{-\beta-1}(A-x)^{-\alpha-1}, \tag{B.15}$$

where $\alpha = \frac{2b}{\sigma^2} - \frac{2a}{\sigma^2 A} - 1$ and $\beta = \frac{2a}{\sigma^2 A} - 1$.

- Boundary behavior for the Jacobi process is the same as for CIR process at the left boundary:

$$D^1 = \begin{cases} \text{exit, if } \beta \le -1 \\ \text{regular, if } -1 < \beta < 0 \\ \text{entrance, if } 0 \le \beta \end{cases} \tag{B.16}$$

The same classification applies to right boundary, we only need to replace β by α. Notice that in the case when $a > 0$, $b > 0$ and $\frac{a}{b} < A$ (which means that mean-reverting level lies in the interval $(0, A)$), we have $\alpha > -1$ and $\beta > -1$ and thus both boundaries are not exit.

- Spectrum of the generator:

$$\lambda_n = -\frac{\sigma^2}{2} n(n - 1 + \frac{2b}{\sigma^2}). \tag{B.17}$$

- Eigenfunctions of the generator:

$$\psi_n(x) = P_n^{(\alpha,\beta)}(y), \tag{B.18}$$

where $y = (\frac{2x}{A} - 1)$ and $P_n^{(\alpha,\beta)}(y)$ are Jacobi polynomials with the three term recurrence relation:

$$\begin{aligned} yP_n^{(\alpha,\beta)}(y) = \quad & \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} P_{n+1}^{(\alpha,\beta)}(y) + \\ & + \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} P_n^{(\alpha,\beta)}(y) + \\ & + \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} P_{n-1}^{(\alpha,\beta)}(y). \end{aligned}$$

- Orthogonality relation

$$\int_D \psi_n(x)\psi_m(x)\nu(x)dx = p_n^2\delta_{nm} = \frac{(\alpha+1)_n(\beta+1)_n}{(\alpha+\beta+2)_{n-1}(2n+\alpha+\beta+1)n!}\delta_{nm}.$$

- Eigenfunction expansion of the probability function:

$$p_t^{(\text{Jacobi})}(x_0, x_1) = \sum_{n=0}^{\infty} \frac{e^{-\lambda_n t}}{p_n^2} P_n^{(\alpha,\beta)}(y_0) P_n^{(\alpha,\beta)}(y_1)\nu(y_1), \tag{B.19}$$

where $y_i = (\frac{2x_i}{A} - 1)$.

Appendix B.4. **Hahn process.**

- Generator

$$\mathcal{L} = (a - bx)\nabla_\delta + \frac{\sigma^2}{2} x(A - x)\Delta_\delta. \tag{B.20}$$

- Domain $D = \{0, \delta, 2\delta, \dots, N\delta\}$
- Spectrum of the generator:

$$\lambda_n = -\frac{\sigma^2}{2} n(n - 1 + \frac{2b}{\sigma^2}). \tag{B.21}$$

- Eigenfunctions of the generator: Hahn polynomials

$$\psi_n(x) = Q_n(x/\delta; \alpha, \beta, N), \tag{B.22}$$

where the parameters α, β and N can be found from the following system of equations:

$$\begin{cases} \beta + \alpha + 1 = \frac{2b}{\sigma^2} - 1, \\ N + \beta + 1 = \frac{A}{\delta}, \\ N(\alpha + 1) = \frac{2a}{\sigma^2 \delta}, \\ N \in \mathbb{N}. \end{cases} \tag{B.23}$$

Note that due to the restriction $N \in \mathbb{N}$, these equations do not have solutions for all values of parameters a, b, A, σ. In applications we start with parameters β, α, N and then compute a, b, A, σ.

Hahn polynomials satisfy the following three terms recurrence relation:

$$-yQ_n(y) = A_n Q_{n+1}(y) - (A_n + C_n)Q_n(y) + C_n Q_{n-1}(y), \tag{B.24}$$

where $Q_n(y) = Q_n(y; \alpha, \beta, N)$ and

$$\begin{cases} A_n = \frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \\ C_n = \frac{n(n+\alpha+\beta+N+1)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}. \end{cases}$$

- Orthogonality measure is given by the scaled *hypergeometric distribution*:

$$\nu(\{\delta y\}) = \begin{pmatrix} \alpha + y \\ y \end{pmatrix} \begin{pmatrix} \beta + N - y \\ N - y \end{pmatrix} \tag{B.25}$$

Orthogonality relation:

$$\sum_{y=0}^{N} Q_n(y; \alpha, \beta, N) Q_m(y; \alpha, \beta, N)\nu(\{y\delta\}) = q_n^2 \delta_{nm},$$

$$q_n^2 = \frac{(-1)^n (n + \alpha + \beta + 1)_{N+1}(\beta + 1)_n n!}{(2n + \alpha + \beta + 1)(\alpha + 1)_n(-N)_n N!}.$$

- Eigenfunction expansion of the probability function:

$$p_t^{(\text{Hahn})}(x_0, x_1) = \sum_{n=0}^{\infty} \frac{e^{-\lambda_n t}}{q_n^2} Q_n(y_0; \alpha, \beta, N) Q_n(y_1; \alpha, \beta, N)\nu(\{y_1\}), \tag{B.26}$$

where $y_i = x_i/\delta$.

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, LONDON, SW7 2AZ, UK
E-mail address: claudio.albanese@imperial.ac.uk

ALEXEY KUZNETSOV, DEPARTMENT OF MATHEMATICS AND STATISTICS, MCMASTER UNIVERSITY,HAMILTON, CANADA
E-mail address: kuznets@math.mcmaster.ca