

# King's College London

UNIVERSITY OF LONDON

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**Candidate No:** ..... **Desk No:** .....

MSC EXAMINATION

CMFM03 FINANCIAL MARKETS

SUMMER 2008

TIME ALLOWED: TWO HOURS

FULL MARKS WILL BE AWARDED FOR COMPLETE ANSWERS TO FOUR QUESTIONS. ONLY THE BEST FOUR QUESTIONS WILL COUNT TOWARDS GRADES A AND B, BUT CREDIT WILL BE GIVEN FOR ALL WORK DONE FOR LOWER GRADES.

WITHIN A GIVEN QUESTION, THE RELATIVE WEIGHTS OF THE DIFFERENT PARTS ARE INDICATED BY A PERCENTAGE FIGURE.

IMPORTANT, PLEASE READ: In the case of numerical answers, a concise numerical formula will suffice. For example,  $x = \frac{1}{2}(3 + 8)$  instead of  $x = 5.5$  will receive full marks.

NO CALCULATORS ARE PERMITTED.

**TURN OVER WHEN INSTRUCTED**

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1. This is a question about interest rates, bond pricing, and bond futures.

- (a) Semiannual compounded forward rates for the next 4 semesters are 1%, 1.5%, 2% and 2.2%. Evaluate the discount curve at maturities of 6, 12, 18 and 24 months. What is the forward price of the one-into-one year zero coupon bond? [20%]

Discount factors are

$$Z(6m) = \frac{1}{1 + 0.5 \cdot 1\%}, \quad (1)$$

$$Z(1y) = Z(6m) \cdot \frac{1}{1 + 0.5 \cdot 1.5\%}, \quad (2)$$

$$Z(18m) = Z(1y) \cdot \frac{1}{1 + 0.5 \cdot 2\%}, \quad (3)$$

$$Z(2y) = Z(18m) \cdot \frac{1}{1 + 0.5 \cdot 2.2\%}, \quad (4)$$

The forward price of a one-into-one year zero coupon bond is

$$F(1y, 1y) = \frac{Z(2y)}{Z(1y)}, \quad (5)$$

- (b) Consider a two-year coupon bond paying a semi-annual coupon in addition to a notional repayment at maturity. [45%]

(i) What should the coupon amount be for the bond to trade at par?

The PV of a 2-year bond of coupon amount  $C$  and nominal  $N$  is

$$PV = C \cdot (Z(6m) + Z(1y) + Z(18m) + Z(2y)) + N \cdot Z(2y). \quad (6)$$

In order for the bond to trade at par, the coupon amount needs to be equal to

$$C = N \frac{1 - Z(2y)}{Z(6m) + Z(1y) + Z(18m) + Z(2y)}. \quad (7)$$

(ii) Assuming the bond trades at par, what are its yield to maturity and its Macaulay duration?

The yield to maturity  $Y$  for a bond trading at par solves the equation:

$$N = PV(Y) \quad (8)$$

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where

$$PV(Y) = \frac{1}{(1 + 0.5Y)^4} + \frac{1 - Z(2y)}{Z(6m) + Z(1y) + Z(18m) + Z(2y)} \sum_{m=1,..,4} \frac{1}{(1 + 0.5Y)^m}. \quad (9)$$

The bond Macauly duration is

$$D = 2 \frac{1}{(1 + 0.5Y)^4} + \frac{1 - Z(2y)}{Z(6m) + Z(1y) + Z(18m) + Z(2y)} \sum_{m=1,..,4} \frac{0.5 \cdot m}{(1 + 0.5Y)^m}. \quad (10)$$

(iii) Consider a bond initially at par. If the yield of the bond moves up by 10bp, how does the price change? How does this rate of change relate to the Macauly duration?

The sensitivity of the bond PV on the yield bond is given by the dollar duration

$$-\frac{\partial PV(Y)}{\partial Y} = \frac{2}{(1 + 0.5 \cdot Y)^5} + \frac{1 - Z(2y)}{Z(6m) + Z(1y) + Z(18m) + Z(2y)} \sum_{m=1,..,4} \frac{0.5 \cdot m}{(1 + 0.5Y)^{m+1}}. \quad (11)$$

- (c) Explain why the future's price of a one-in-two year zero coupon bond is not equal to the forward price. Is the future price above or below the forward price? [35%]

A bond futures contract is equivalent to a series of daily forward contracts, each one being reset on a daily basis and delivering the following day contract at equilibrium, except for the contract on the last day which delivers the bond. The daily resets result in daily cash flows. The future price process  $f_t$  is a martingale under the risk neutral measure, i.e.

$$f_t = E_t[e^{-\int_t^{T_1} r_s ds} Z_{T_1}(T_2)]. \quad (12)$$

The forward price instead is

$$F_t = \frac{Z_t(T_2)}{Z_t(T_1)}. \quad (13)$$

The two are equal if interest rates are deterministic. Instead, if interest rates are stochastic, since the processes for the discount factors  $Z_t(T_2)$  and  $Z_t(T_1)$  are typically positively correlated, the drift for forward prices is typically negative. Hence, forward bond prices are typically higher than futures bond prices.

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**End of Question 1**

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2. This is a question about variance swaps and trading strategies in futures. The payoff of a variance swap is defined as  $RV - SR^2$  where  $SR$  is a constant expressed in units of volatility and  $RV$  is the realized variance of arithmetic returns given by:

$$RV = \frac{1}{N} \sum_{i=0}^{N-1} \left( \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \right)^2. \quad (14)$$

- (a) A common assumption when valuing variance swaps is that returns are sufficiently small that the Taylor expansion for a log-return can be truncated to the second order, i.e.:

$$\log \frac{S_{t_{i+1}}}{S_{t_i}} = \log \left( \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} + 1 \right) \approx \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} - \frac{1}{2} \left( \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \right)^2. \quad (15)$$

On March 2nd 2009 the S&P500 index fell 35 points to 700. What was the relative error in the formula for the log-return on that day? [10%]

The error was

$$\log \frac{700}{735} - \frac{35}{735} + \frac{1}{2} \left( \frac{35}{735} \right)^2 \approx \frac{1}{6} \left( \frac{35}{735} \right)^3 \approx 2 \cdot 10^{-5}. \quad (16)$$

- (b) Based on the Taylor expansion for log-returns and assuming that interest rates are zero, show that realized variance can be expressed as follows:

$$RV \approx \frac{1}{N} \sum_{i=0}^{N-1} \left( \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \right)^2 \approx -\frac{2}{N} \sum_{i=0}^{N-1} \log \frac{S_{t_{i+1}}}{S_{t_i}} + \frac{2}{N} \sum_{i=0}^{N-1} \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \quad (17)$$

[10%]

It suffices to sum up and equate both sides in the Taylor expansion above in (15), i.e.

$$\frac{1}{N} \sum_{i=0}^{N-1} \log \frac{S_{t_{i+1}}}{S_{t_i}} \approx \frac{1}{N} \sum_{i=0}^{N-1} \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} - \frac{1}{2} \frac{1}{N} \sum_{i=0}^{N-1} \left( \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \right)^2. \quad (18)$$

- (c) Assume interest rates are zero. Explain how the sum

$$\frac{2}{N} \sum_{i=0}^{N-1} \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}. \quad (19)$$

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can be obtained by means of a strategy in futures. What is the cost of this strategy? [40%]

In case interest rates are zero, the strategy in futures consists in taking a position in the stock in  $\frac{1}{S_{t_i}}$  on the  $i$ -th day and holding it for one day only. If interest rates are zero, the futures price over one day is equal to the stock spot price and hence the cost of this strategy is zero.

- (d) Assume that interest rates are at 4.5% and the variance swap maturity is one year. How should one modify the strategy in futures to reproduce the summation in (19)? What is the cost of the trading strategy in futures in this case, accounting for the cost of funding? [40%]

If interest rates are not zero, the futures price over one day is equal to

$$f_{t_i} = (1 + \delta tr)S_{t_i} \quad (20)$$

where  $\delta t = \frac{1}{362.25}$  and  $r = 4.5\%$ . The strategy of holding  $\frac{1}{S_{t_i}}$  futures contracts on the  $i$ -th day for one day only yields a non-zero payout equal to

$$\frac{2}{N} \sum_{i=0}^{N-1} \frac{S_{t_{i+1}} - (1 + \delta tr)S_{t_i}}{S_{t_i}} \quad (21)$$

and has zero present value. Hence the present value of

$$\frac{2}{N} \sum_{i=0}^{N-1} \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \quad (22)$$

is  $\frac{2rT}{N}$ .

**End of Question 2**

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3. This is a question about interest rate swaps and floating rate notes.

- (a) Define the LIBOR forward curve, the zero curve and the LIBOR swap curve. Compare the usual relative smoothness of each curve. [25%]

Let  $Z_t(T)$  be the discount factor for the maturity  $T$  as observed at time  $t$ . The zero curve continuous compounding is defined as the curve  $y_t(T)$  such that

$$Z_t(T) = \exp(- (T - t)y_t(T)). \quad (23)$$

The LIBOR forward curve with simple compounding of period  $\tau$  is the curve  $L_t(T_i)$  where  $T_i = i \cdot \tau$  such that

$$\frac{1}{1 + \tau L_t(T_i)} = \frac{Z_t(T_i + \tau)}{Z_t(T_i)}. \quad (24)$$

The swap curve  $SR_t(T_i)$  where  $T_i = i \cdot \tau$  and  $\tau$  is a simple compounding period is defined as follows:

$$SR_t(T_i) = \frac{Z_t(T_i) - Z_t(0)}{\sum_{j=1}^i \tau Z_t(T_j)}. \quad (25)$$

Typically, the swap curve is smoother than the zero curve and the zero curve is smoother than the forward curve.

- (b) Is it possible that the 20 year swap rate is greater than the 2 year swap rate? Can the spread  $s_{20} - s_2$  be arbitrarily large? Is it possible that the 2 year swap rate is greater than the 20 year? Can the spread  $s_2 - s_{20}$  be arbitrarily large? [25%]

Yes and yes, i.e. neither inequality contradicts the assumption of absence of arbitrage or of non-negative rates. In fact, suppose that interest rates are 10% for two years and thereafter they are identically zero. Then the 2 year swap rate is higher than the 10 year swap rate. The same argument shows that the spread  $s_{20} - s_2$  can be arbitrarily large.

Conversely, suppose that the interest rate is identically zero for two years and then equal to 10% thereafter. Then the 20 year swap rate is higher than the 2 year swap rate. However, the spread  $s_2 - s_{20}$  cannot be arbitrarily large as this would imply that some forward rates are negative.

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- (c) Consider a floating rate note paying twice LIBOR on a semiannual basis plus the notional repayment at maturity. On what rate does the price of the floating rate note depend on a coupon date? How does the answer change in between coupon dates? [25%]

The price of the floating rate note at a cash flow date is not subject to interest rate risk. The price in between two cash flow dates depends on the rate corresponding to the maturity given by the next cash flow date.

- (d) Consider a forward starting swap, with start date in 2 years and tenor 10 years. Give a formula for the equilibrium swap rate at current time and provide a derivation. [25%]

Assuming to be specific that the coupon frequency is semiannual, the formula is

$$SR_t = \frac{Z_t(2y) - Z_t(10y)}{\sum_{j=5}^{20} \tau Z_t(T_j)}. \quad (26)$$

**End of Question 3**

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4. This is a question about equity options.

- (a) Assuming no dividends and that the stock cannot default, can one replicate a forward contract at a given maturity by means of a position in a call and a position in a put options? How does the answer change if the stock pays dividends? How does the answer change if one allows for the possibility that the worth of the stock price vanishes prior to maturity due to default? [30%]

In all situations, a forward contract is replicated by a long position in a call struck at the forward price combined with a short position in a put struck at the forward price. The possibility of dividends and default events do not affect the conclusion.

- (b) Explain how to replicate a futures contract on a non-dividend paying stock. What is the cost of replication? How does the answer change if the stock pays dividends? [30%]

To replicate a long position in a futures contract on a non-dividend paying stock one can buy the stock funding the purchase with a stream of one-day loans, while depositing in a cash account the difference between the stock price realized on each day and the corresponding futures price on the previous day.

- (c) Consider the capped logarithmic payoff of European type

$$\min \left( -\log \left( \frac{S_T}{S_0} \right), \text{Cap} \right) \quad (27)$$

where  $T$  is the maturity. Explain how to replicate this payoff approximately with a strategy involving only out of the money call options, out of the money put options, the stock and cash. [40%]

To replicate this payoff, one would hold an amount equal to the cap in cash, a short position in the stock of size

$$-\frac{1}{S_0}, \quad (28)$$

a short position in a put struck at  $S_0 e^{-\text{Cap}}$  and a continuum of long positions in call and put options at strikes  $K > S_0 e^{-\text{Cap}}$  of (infinitesimal) position  $n(K)dK$  to match the convexity of the log payoff, i.e. such that

$$n(K) = \frac{1}{K^2}. \quad (29)$$

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**End of Question 4**

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5. This is a question about forward and futures contracts.

(a) A short forward contract on gold that was negotiated three months ago will expire in one year and has delivery price of \$980 per ounce. The current price of gold is \$950 per ounce, the annual storage cost is \$0.50 per ounce and the risk-free rate is 3% per year (simply compounded).

(i) Calculate the current equilibrium forward price of gold for delivery in one year.

The equilibrium forward price is

$$(\$950 + \$0.50)(1 + 3\%) \approx 979 \tag{30}$$

(ii) Calculate the current value of the forward contract issued three months ago. [25%]

The value is  $\frac{\$1}{1+3\%}$ .

(b) On December 3rd 2008, at the Comex in New York, December gold futures were in backwardation. December 31 deliveries were quoted at 2% discount to spot, while gold futures with delivery February 27 were quoted at 0.29% discount to spot. (All percentages annualized.) How would you interpret these quotes? [25%]

Since carry costs are positive and interest rates are also positive, the remaining factor that can potentially account for backwardation is the credit risk of the long party promising delivery.

(c) A bank sells forward contracts on oil with delivery in one year to two different counter-parties. The contracts are otherwise identical but the forward price to one counter-party is \$45 per barrel, while the other counterparty pays \$45.5 per barrel. What reasons can motivate the difference? By how much would the futures prices on the same underlying differ among the two counter-parties? [25%]

The two counter-parties may have different credit risk and/or there could be a different correlation between the probability of counter-party default and a rally in oil prices.

(d) Metallgesellschaft used to be a major firm that would sell forward contracts for oil delivery to clients hedging them with futures contracts. In December 1993 the strategy gave rise to mark-to-market losses exceeding 1.5 billions. How could this hedging strategy go so wrong? [25%]

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The hedging strategy could go wrong because the futures positions gives rise to a continuous stream of cash flows. To cover, Metall-Gesellschaft had to take long futures position in the underlying. Since oil prices fell in that period, there were a series of margin calls which deteriorated the cash position of the firm and affected its credit rating, raising as a consequence its financing costs.

**End of Question 5**

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