Financial Markets, Lecture 6

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Markov Processes

Markov processes are a particularly important special class of processes characterized by the fact that transition probability kernels are independent of the past values attained by the path γ. More precisely, elementary transition probability kernels for a Markov process have the special form

$$q(\gamma, t_i) = u_{\delta t}(\gamma_i, \gamma_{i+1}; t_i) \tag{1}$$

where $u_{\delta t}(y_1, y_2; t_i)$ is a function of $y_1, y_2 \in \Lambda$ and time t_i.

Markov Generators

The Markov generator or Markovian is given by the matrix $\mathcal{L}\left(y_1, y_2; t_i\right)$ such that

$$u_{\delta t}(y_1, y_2; t_i) = \delta_{y_1,y_2} + \delta t \mathcal{L}\left(y_1, y_2; t_i\right) \tag{2}$$

for all $y_1, y_2 \in \Lambda$ and all t_i, $i = 0, 1, \ldots$. Here

$$\delta_{y_1,y_2} = \begin{cases} 1 & \text{if } y_1 = y_2 \\ 0 & \text{if } y_1 \neq y_2 \end{cases} \tag{3}$$

is the so called Kronecker Delta.

Markov Generators

The constraints of positivity and probability conservation imply the following two conditions on the matrix $\mathcal{L}(y_1, y_2; t_i)$:

(i) $\mathcal{L}(y_1, y_2; t_i) \geq 0$ for all $y_1 \neq y_2$ and all t_i;

(ii) $\sum_{y_2} \mathcal{L}(y_1, y_2; t_i) = 0$ for all y_1 and all t_i.

Markov Generators

The two conditions above are necessary but not sufficient as, due to condition (i) and (ii), the diagonal matrix elements

$$\mathcal{L}(y_1, y_1; t_i) = -\sum_{y_2 \neq y_1} \mathcal{L}(y_1, y_2; t_i) \leq 0 \tag{4}$$

are non-positive. To ensure positivity of the diagonal elements of the elementary kernel, we thus need to postulate the following third property called the Courant condition:

(iii) $\delta t \leq \frac{1}{\max_y |\mathcal{L}(y, y; t_i)|}$.

Markov Generators

Definition

A matrix $\mathcal{L}(y_1, y_2; t_i)$ satisfying the properties (i), (ii) and (iii) above is called Markov matrix or Markovian.

In practice, one builds elementary transition probability kernels starting from a Markovian, given which one finds the elementary time interval $\delta t > 0$ (typically one day or a fraction of a day) satisfying the Courant condition.

Operators and Kernels

As a matter of terminology, we distinguish between operators and kernels. A kernel is a matrix $A(x, y)$ with indices x, y taking up a finite set of values. In the previous example $x, y = 0, ...d - 1$. A vector is instead represented by an array $v(x)$. The matrix $A(x, y)$ can also be put in relation with an operator A that transforms vectors linearly, so that

$$(Av)(x) = \sum_y A(x, y)v(y). \tag{5}$$

Vice versa, to each operator that transforms vectors linearly there corresponds a matrix. In fact, if one considers the vector $\delta_y(x) = \delta(x - y)$, one finds

$$(A\delta_y)(x) = A(x, y). \tag{6}$$

Given an operator A, the matrix $A(x, y)$ is called the *kernel of A*.

Transition Probability Kernels

A *transition probability kernel* over finite time intervals is given by a two-parameter family of matrices $u\left(y_1, t_1; y_2, t_2\right)$ dependent on the time coordinates $t_1 \leq t_2$ and representing the transition probabilities from state y_1 at time t_1 to state y_2 at time t_2. For fixed t_1 and t_2, these are the transition probability kernels. The operator corresponding to a transition probability kernel is called *propagator*.

The incidence matrix κ is characterized by the set of all admissible set $\mathcal{P}(\Lambda, \kappa)$. In turn, admissible paths are characterized by the condition

$$u\left(\gamma_i, t_i; \gamma_{i+1}, t_{i+1}\right) > 0. \tag{7}$$

being satisfied for all $i \geq 0$.

Transition Probability Kernels

Given the family of elementary transition probability kernels $u_{\delta t}(y_1, y_2; t_i)$ for a Markov process, one can compute more general transition probability kernels $u(y_1, t_i; y_2, t_j)$ over arbitrary time intervals $[t_i, t_j]$ with $i < j$. In the particular case of a time step twice the size of δt from the equation (**??**), we find

$$u(\gamma_i, t_i; \gamma_{i+2}, t_{i+2}) = \sum_{\gamma_{i+1}} u_{\delta t}(\gamma_i, \gamma_{i+1}; t_i)\, u_{\delta t}(\gamma_{i+1}, \gamma_{i+2}; t_{i+1}) \quad (8)$$

Remarkably, this law is the same as the standard rule for multiplying matrices rows by columns. This is perhaps the single most noteworthy property of Markov processes which allows one to reduce problems in probability theory to linear algebra.

Transition Probability Kernels

In matrix language, the equation above can be recast as follows:

$$u\left(t_i; t_{i+2}\right) = u_{\delta t}\left(t_i\right) u_{\delta t}\left(t_{i+1}\right). \tag{9}$$

For two generic time indices $i < j$, we have that

$$u\left(t_i; t_j\right) = u_{\delta t}\left(t_i\right) \cdots u_{\delta t}\left(t_{j-1}\right). \tag{10}$$

Pathwise representation

The path-wise representation for the latter formula is

$$u\left(\gamma_i, t_i; \gamma_j, t_j\right) = \sum_{\gamma:\gamma_i \to \gamma_j} u_{\delta t}\left(\gamma_i, \gamma_{i+1}; t_i\right) \cdots u_{\delta t}\left(\gamma_{j-1}, \gamma_j; t_{j-1}\right) \tag{11}$$

where the sum is over all paths $\gamma = \{\gamma_i, \ldots, \gamma_j\}$, γ_k is a state variable and $k = i, \ldots j$.

Fast Exponentiation

In the important case where the transition probability kernels $u_{\delta t}(x, y; t)$ are time-homogeneous, i.e. independent of time, over a certain time interval, then the numerical evaluation of kernels by matrix products can be greatly sped up by the so called fast exponentiation algorithm. Fast exponentiation proceeds iteratively evaluating first

$$u_{2\delta t} = u_{\delta t} \cdot u_{\delta t}. \tag{12}$$

Fast Exponentiation

Here we suppress writing the arguments x, y as we are writing this equation in operator notation as opposed to in kernel or matrix notation. We also suppress denoting the time coordinate as we are assuming time homogeneity. As a second step one evaluates

$$u_{4\delta t} = u_{2\delta t} \cdot u_{2\delta t} \tag{13}$$

and then iterates until one finds after n steps

$$u_{2^n\delta t} = u_{2^{n-1}\delta t} \cdot u_{2^{n-1}\delta t}. \tag{14}$$

Brownian Motion

Brownian motion can be described as a limit of the discrete process on the lattice $h\mathbb{Z} \equiv \{0, \pm h, \pm 2h, ...\}$ with Markov generator

$$\mathcal{L}(x, y) = \frac{1}{2}\Delta_h(x, y) \tag{15}$$

where $\Delta_h(x, y)$ is the matrix of the discrete Laplace operator given by

$$\Delta_h(x, y) = \frac{\delta_{x+h,y} + \delta_{x-h,y} - 2\delta_{x,y}}{h^2}. \tag{16}$$

Elementary Transition Probability Kernel

The elementary transition probability matrix is given by

$$u_{\delta t}(x, y) = \delta_{xy} + \delta t \mathcal{L}(x, y) \tag{17}$$

where δt satisfies the Courant condition

$$\delta t \leq \frac{1}{\mathcal{L}(x, x)} \equiv h^2. \tag{18}$$

Finite Time Transition Probability Kernel

To numerically evaluate the transition probability matrix over a long time step, one can impose boundary conditions, form a finite matrix and then use the fast exponentiation algorithm. An alternative is to use the analytic approximation below. Until recent years, fast exponentiation was not technically viable and analytic approximations were the only route to compute kernels.

Eigenvalues of the Markovian

We start from observing that a complete set of orthogonal eigenvectors for the matrix $\mathcal{L}(x, y)$ is given by the functions

$$v_k(x) = e^{ikx} \tag{19}$$

Here, $k \in \left[-\frac{\pi}{h}, \frac{\pi}{h} \right]$. In fact, the following eigenvalue equation is satisfied:

$$\mathcal{L} v_k = \left(\frac{\cos kh - 1}{h^2} \right) v_k \tag{20}$$

Fourier Series

Furthermore, according to the theory of Fourier series, all lattice functions $f(x)$ with $\sum_x |f(x)^2| < \infty$ can be written as follows

$$f(x) = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \hat{f}(k) e^{-ikx} \frac{dk}{2\pi} \tag{21}$$

where

$$\hat{f}(k) = h \sum_{-\infty}^{\infty} f(x) e^{ikx} \tag{22}$$

Fourier Series

Because of the eigenvalue equation in (20), we have that

$$u_{\delta t} v_k = \left(1 + \delta t \frac{\cos kh - 1}{h^2}\right) v_k . \tag{23}$$

The N-th power of this matrix thus satisfies the equation

$$u_{N\delta t} v_k = u_{\delta t}^N v_k = \left(1 + \delta t \frac{\cos kh - 1}{h^2}\right)^N v_k . \tag{24}$$

Translation Invariance

Notice that the matrix elements $\mathcal{L}(x, y)$ depend only on the difference $x - y$. This property is also shared by $u_{N\delta t}$ whose matrix elements can be written as follows

$$u_{N\delta t}(x, y) = U_{N\delta t}(x - y). \tag{25}$$

Hence

$$\sum_y U_{N\delta t}(x - y)e^{iky} = \left(1 + \delta t \frac{\cos kh - 1}{h^2}\right)^N e^{ikx}. \tag{26}$$

Translation Invariance

Otherwise stated

$$\sum_y U_{N\delta t}(x-y)e^{-ik(y-x)} = \left(1+\delta t\frac{\cos kh - 1}{h^2}\right)^N . \tag{27}$$

The Fourier inversion formula yields the density

$$h^{-1}U_{N\delta t}(x-y) = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left(1+\delta t\frac{\cos kh - 1}{h^2}\right)^N e^{ik(y-x)}\frac{dk}{2\pi} \tag{28}$$

Taking the Limit

To fulfill the Courant condition, let us set $\delta t = h^2$. We intend to take a limit as $h \to 0$ while holding $T = \delta t N$ constant. Hence N depends on h so that

$$N = N(h) = T h^{-2}. \tag{29}$$

By expanding the integrand in powers of h we find

$$h^{-1} U_T(x-y) = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left(1 - \frac{T}{N(h)} \frac{k^2}{2} + O(h^2)\right)^{N(h)} e^{ik(y-x)} \frac{dk}{2\pi} \tag{30}$$

Taking the Limit

Taking the limit as $h \longrightarrow 0$ and applying Neper's formula we arrive at the following equation for the limit probability density

$$p_T(x,y) = \lim_{h\to 0} h^{-1} U_T(x-y) \tag{31}$$
$$= \lim_{h\to 0} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-\frac{Tk^2}{2}} e^{ik(y-x)} \frac{dk}{2\pi} = \frac{1}{\sqrt{2\pi T}} e^{-\frac{(x-y)^2}{2T}}. \tag{32}$$

This density gives the transition probability for Brownian motion.

Transition Probability Kernel

Notice that the time coordinate t is dimensionless in this equation. In practice, time is measure in some physical units (typically years in Finance). It is thus necessary to accompany calendar time with a factor that transforms it in a pure number. Typically, one writes this factor as the square of a parameter σ, so that $\sigma^2 T$ is dimensionless. The transition probability for Brownian motion with dimensional time is thus written as follows:

$$\begin{aligned} p_T(x, y) \qquad &= \lim_{h\to 0} h^{-1} U_{\sigma^2 T}(x-y) && (33) \\ &= \lim_{h\to 0} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-\frac{\sigma^2 T k^2}{2}} e^{ik(y-x)} \frac{dk}{2\pi} && (34) \\ &= \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{(x-y)^2}{2\sigma^2 T}}. && (35) \end{aligned}$$

Notations

Let us introduce the notations

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{36}$$

for the normal distribution density function and

$$\Phi(x) = \int_{-\infty}^{x} \phi(y) dy \tag{37}$$

for the cumulative normal distribution function. Let us notice that $\lim_{x\to-\infty} \Phi(x) = 0$ and $\lim_{x\to+\infty} \Phi(x) = 1$.

Martingale Condition

We have that

$$E_t[x_T | x_t = x] = \frac{1}{\sigma\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\sigma^2 T}} y dy = x. \qquad (38)$$

Hence, Brownian motion is a martingale.

Martingale Condition

We have

$$E_t[\exp(x_T)] = \exp\left(\frac{x + \sigma^2 T}{2}\right). \tag{39}$$

Hence, also the process

$$X_t = x_t e^{\frac{-\sigma^2 t}{2}} \tag{40}$$

is a martingale. The process following by X_t is called *log-normal*.

Second moments

We also have that

$$E_t[(x_T - x_t)^2 | x_t = x] = \frac{1}{\sigma\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\sigma^2 T}} (y-x)^2 dy = \sigma^2 T. \tag{41}$$

Hence, the standard deviation at time T of the distribution of Brownian motion of volatility σ is $\sigma\sqrt{T}$. A similar formula for the log-normal process is

$$E_t[(X_T - X_t)^2 | X_t = X] = \frac{1}{\sigma\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\sigma^2 T}} (e^{y-\frac{\sigma^2 T}{2}} - e^x)^2 dy = e^{2\sigma^2 T} \tag{42}$$

Bachelier and Black-Scholes Formulas

The equation

$$E_t[(x_T - K)_+ | x_t = x] = (x - K)\Phi\left(\frac{x - K}{x\sigma\sqrt{T}}\right) + x\sigma\sqrt{T}\phi\left(\frac{x - K}{x\sigma\sqrt{T}}\right) \tag{43}$$

is called *Bachelier formula*. The equation

$$E_t[(X_T - K)_+ | X_t = X] = X\Phi\left(\frac{\log\frac{X}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{X}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \tag{44}$$

is called the *Black-Scholes formula*.

Bachelier and Black-Scholes Formulas

The Bachelier formula was introduced by Bachelier in 1900 in his PhD thesis and is currently used in as a convention for price quoting. The Bachelier model applies to situations where one is interested in evaluating the expectation of a stochastic process which is known to be a martingale. If this process is modeled as Brownian motion, the Bachelier formula provides a closed form expression for call option prices. If instead one postulates that the martingale process is a log-normal martingale, one obtains the Black-Scholes formula for call options.

Bachelier and Black-Scholes Formulas

Applications of the Bachelier and Black-Scholes formula in equity markets apply to various payoffs including call options of payoff $(S_T - K)_+$. Here, S_t denotes the stock price process, T is the maturity, K is the strike. To setup a Bachelier model we need to consider a restricted universe in which only the option, the stock price S_t and a numeraire asset g_t are traded and then apply the Fundamental Theorem to this ideal situation. This approach is called *local calibration*. Regarding the numeraire asset, two possibilities are convenient: the forward measure of numeraire $g_t = Z_t(T)$ and the risk neutral measure where the numeraire is the money-market account.

Forward Measure

To use the forward measure, consider the forward price process

$$F_t(T) = \frac{S_t}{Z_t(T)}. \tag{45}$$

According to the Fundamental Theorem, we require that

$$F_t(T) = E_t^{Z(T)}[S_T]. \tag{46}$$

because $S_T = F_T(T)$.

Bachelier Formula

To achieve this condition, we model the forward as a Brownian motion, i.e. set $F_t(T) = x_t$, so that the price of a call option can be evaluated as follows by means of a Bachelier formula

$$\begin{aligned} C_t(T,K) &= E_t^{Z(T)}\left[(S_T - K)_+ | F_t(T) = F\right] && (47) \\ &= (F-K)\Phi\left(\frac{F-K}{F\sigma\sqrt{T}}\right) + F\sigma\sqrt{T}\phi\left(\frac{F-K}{F\sigma\sqrt{T}}\right) && (48) \end{aligned}$$

Black-Scholes Formula

We can also set $F_t(T) = X_t$ and in this case we arrive at the Black-Scholes formula

$$\begin{aligned} C_t(T,K) &= E_t^{Z_t(T)}[(S_T - K)_+ | F_t(T) = F] \quad (49) \\ &= F\Phi\left(\frac{\log\frac{F}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{F}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right). \quad (50) \end{aligned}$$

The Bachelier and the Black-Scholes formula are obviously different. However, the differences are numerically minor as long as maturities are short and one uses a Bachelier volatility σ_B equal to $\sigma_{BS} F$, where σ_{BS} is the Black-Scholes volatility.

Risk Neutral Measure: Bachelier Formula

In alternative, one can choose the money-market account B_t as a numeraire and express the Bachelier formula in terms of the futures price process $f_t(T)$ which is a martingale under this choice of measure. In this case, one can consider the case of options traded on a futures basis whereby the price changes of option contracts are settled on a daily basis, similarly to what happens for futures contracts on the underlying. The future price process for call options is also a martingale under the risk neutral measure. The Bachelier formula for futures price of call options reads

$$\begin{aligned} c_t(T,K) &= E_t^{Z(T)}\left[(S_T - K)_+ | f_t(T) = f\right] && (51) \\ &= (f-K)\Phi\left(\frac{f-K}{f\sigma\sqrt{T}}\right) + f\sigma\sqrt{T}\phi\left(\frac{f-K}{f\sigma\sqrt{T}}\right) && (52) \end{aligned}$$

Risk Neutral Measure: Black-Scholes Formula

We can also set $f_t(T) = X_t$ and in this case we arrive at the Black-Scholes formula for the future price of call options is

$$\begin{aligned} c_t(T, K) &= E_t^{B_t}[(S_T - K)_+ | f_t(T) = f] && (53) \\ &= f\Phi\left(\frac{\log\frac{f}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{f}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right). && (54) \end{aligned}$$

Since the forward and the futures price of a stock are in general different, one arrives to slightly different prices for the same choice of volatility.

Use of the Bachelier and Black-Scholes Formula

The Bachelier and Black-Scholes formula are commonly used also as quoting conventions. The idea is that if one fixes the price, then there is a volatility which reproduces it. That is called the *implied* Bachelier or Black-Scholes volatility.

The Bachelier and Black-Scholes local calibration procedure also extend to interest rate swaptions. In this case, the procedure involves identifying a small ideal market in which the only traded contracts are: a fixed rate annuity and a floating rate annuity with the same start and end date, and a swaption whose maturity is the start date of the annuities and whose tenor is the difference between their end and start date. Assuming that this small market niche is isolated from the rest of the world, one sets up a model which conforms to the Fundamental Theorem. Of course, if one wishes to include other swaptions of different strike, different maturity or different tenor, then the construction will be repeated

Pricing Swaptions

To price swaptions, consider that the equilibrium swap rate at time t is give by

$$SR_t = \frac{Z_t(T) - Z_t(T_0)}{\sum_{i=0}^{N} \tau Z_t(T_i)} \tag{55}$$

where T_0 is the start date of the swap and we assume that $t < T_0$. A swaption has a payoff equal to

$$(SR_t - \kappa)_+ \sum_{i=0}^{N} \tau Z_t(T_i) \tag{56}$$

at maturity T_0. If we select the numeraire asset $g_t = \sum_{i=0}^{N} \tau Z_t(T_i)$ and consider the asset price process for the swaption SO_t of strike κ, we find

$$SO_t = g_t E_t^g \left[(SR_t - \kappa)_+ | SR_t = r \right]. \tag{57}$$

Pricing Swaptions

Postulating that the swap rate follows a Brownian motion, we can evaluate the expectation as follows by means of the Bachelier formula

$$SO_t = g_t E_t^g \left[(SR_t - \kappa)_+ | SR_t = r \right] \tag{58}$$

$$= g_t (r - \kappa) \Phi \left(\frac{r - \kappa}{r \sigma \sqrt{T}} \right) + g_t r \sigma \sqrt{T} \phi \left(\frac{r - \kappa}{r \sigma \sqrt{T}} \right) \tag{59}$$

If instead we set $SR_t = X_t$, we arrive at the Black-Scholes formula for swaptions

$$SO_t = g_t E_t^g \left[(SR_t - \kappa)_+ | SR_t = r \right] \tag{60}$$

$$= g_t r \Phi \left(\frac{\log \frac{r}{\kappa} + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \right) - g_t \kappa \Phi \left(\frac{\log \frac{r}{\kappa} + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \right). \tag{61}$$