Financial Markets, Lecture 5

Claudio Albanese

King's College London, October 19th, 2009

Fundamental Theorem

Consider a finite lattice Λ and a finite time-step δt_n. Let $\mathcal{P}(\Lambda, k)$ be a pathspace characterized by the incidence matrices $\kappa(\gamma, t_i)$ and let $g(\gamma, t_i)$ be a numeraire asset price process.
Let $A^s(\gamma, t_i)$ be a family of g-coherent non-anticipatory path functionals indexed by $s = 1, ...M$ with $M > 0$ on the time interval $t_i \in [t_0, t_j]$ where $j > 0$.
Let Q be any probability measure with respect to which the processes A^s_n are g-discounted martingales. The existence of such a measure is assured by the Fundamental Theorem of Finance in the previous section.

Contingent Cash Flow Streams

Definition

A **contingent cash flow stream** is given by a non-anticipatory path functional $c(\gamma, t)$ and describes unit of accounts payable by one party to another at time t in case the path γ is realized up to time t.

A special case of a contingent cash flow stream is given by a single payment equal to one unit of account at a fixed maturity date T.

Discounted expectations

A contract that entitles to a payment of amount $c(\gamma, T)$ at time T and zero at all other times is worth $c(\gamma, T)$ units at time T. Hence, applying the Fundamental Theorem, the worth of this contract at times prior to T is given by

$$A(\gamma, t; T) = E_t^q \left[\frac{g(\gamma, t_i)}{g(\gamma, T)} c(\gamma, T) \,\middle|\, \{\gamma_k\}_{k \leq i} \right] \tag{1}$$

$$= \sum_{\gamma' : \gamma'_k = \gamma_k \ \forall k \leq i} q(\gamma', t_i) \quad \dots \quad q(\gamma', T - \delta t) \ \frac{g(\gamma', t_i)}{g(\gamma', T)} c(\gamma', T). \tag{2}$$

Pricing Cash Flow Streams

A contract which entitles to a generic cash flow stream $c(\gamma, t')$ at all times $t' \geq t$ is equivalent to the portfolio of all elementary contracts which entitle to a cash flow of the same amount $c(\gamma, t')$ at one single date in the future. In fact, the difference between the cash flow stream contract and the corresponding portfolio entitles to no cash flows at any time and is thus worth zero. Hence, the price of a cash flow stream $c(\gamma, t')$ for $t' > t$ is given by

$$E_t^q \left[\sum_{t' > t} \frac{g(\gamma, t)}{g(\gamma, t')} c(\gamma, t') \;\middle|\; \{\gamma_k\}_{k \leq i} \right]. \tag{3}$$

Dynamic Trading Strategies

Definition

A **dynamic trading strategy** is given by a family of M non-anticipatory path functionals $\zeta^s(\gamma, t)$ expressing positions in the assets $A^s(\gamma, t_i)$.

Notice that in general a trading strategy will be costly to implement as transactions are required to rebalance a position when needed. The rebalancing cost at time t is given by

$$\sum_{s=1}^{M} (\zeta^s(\gamma, t) - \zeta^s(\gamma, t - \delta t)) A^s(\gamma, t) \tag{4}$$

Cost of Implementing a Dynamic Trading Strategies

Hence, the cumulative cost at time t to implement a trading strategy $\zeta^s(\gamma, t)$, including all past incurred costs and the worth of future ones, is given by

$$D(\gamma, t; \zeta) = E_t^q \left[\sum_{t'>t} \frac{g(\gamma,t)}{g(\gamma,t')} (\zeta^s(\gamma, t') - \zeta^s(\gamma, t' - \delta t)) A^s(\gamma, t') \,\middle|\, \{\gamma_{t''}\right. + \sum_{t' \leq t} \frac{g(\gamma,t)}{g(\gamma,t')} (\zeta^s(\gamma, t') - \zeta^s(\gamma, t' - \delta t)) A^s(\gamma, t').$$

Notice that the cost process of a trading strategy $D(\gamma, t; \zeta)$ is itself a non-anticipatory functional which, in virtue of the Fundamental Theorem, is coherent with respect to the base asset price processes $A^s(\gamma, t)$.

Self-Financing Strategies

Definition

A dynamic trading strategy is called **self-financing** if

$$\sum_{s=1}^{M}(\zeta^s(\gamma, t + \delta t) - \zeta^s(\gamma, t))A^s(\gamma, t + \delta t) = 0 \tag{7}$$

for all paths γ and all times t. The financial meaning of this condition is that the cost for updating a position according to the given strategy from a generic time t to the following instant $t + \delta t$ is zero.

Self-Financing Strategies as Asset Value Processes

Theorem

If $\zeta^s(\gamma, t)$ is a self-financing trading strategy, then if one adds the portfolio value process

$$\Pi(\gamma, t) \equiv \sum_{s=1}^{M} \zeta^s(\gamma, t) A^s(\gamma, t) \tag{8}$$

to the family of asset price processes $A^s(\gamma, t)$, the extended family is still g-coherent.

Self-Financing Strategies as Asset Value Processes

Proof.

Consider the extended family of $M+1$ assets where $A^{M+1}(\gamma,t)=\Pi(\gamma,t)$ and consider the elementary time step from time t to time $t+\delta t$. Let $\xi_s, s=1,...M+1$ be a position vector and assume that it is possible to have

$$\frac{1}{g^s(\gamma,t_{i+1})}\sum_{s=1}^{M+1}\xi^s A^s(\gamma,t_{i+1})-\frac{1}{g^s(\gamma,t_i)}\sum_{s=1}^{M+1}\xi^s A^s(\gamma,t_i)>0. \quad (9)$$

Self-Financing Strategies as Asset Value Processes

Since we are assuming self-financing, we have that

$$A^{M+1}(\gamma, t+\delta t) = \Pi(\gamma, t+\delta t) = \sum_{s=1}^{M} \zeta^s(\gamma, t+\delta t) A^s(\gamma, t+\delta t) = \sum_{s=1}^{M} \zeta^s(\gamma, t \tag{10}$$

We also have that

$$A^{M+1}(\gamma, t) = \sum_{s=1}^{M} \zeta^s(\gamma, t) A^s(\gamma, t). \tag{11}$$

Self-Financing Strategies as Asset Value Processes

Inserting these two equations into (9) we find that this reduces to the following condition:

$$\frac{1}{g^s(\gamma, t_{i+1})} \sum_{s=1}^{M} (\xi^s + \zeta^s(\gamma, t)) A^s(\gamma, t_{i+1}) - \frac{1}{g^s(\gamma, t_i)} \sum_{s=1}^{M} (\xi^s + \zeta^s(\gamma, t)) A^s(\gamma, \tag{12}$$

Due to the assumed coherence of the asset price processes $A^s(\gamma, t_i)$, we conclude that also the extended set of asset price processes is coherent.

Market Completeness

Definition

A family of asset price processes $A^s(\gamma, t_i)$ is called complete if it contains also cost processes of all trading strategies and the value process of all self-financing trading strategies.

Futures Contracts

A futures price process is an non-anticipatory functional $f(\gamma, t)$ such that, at any time t, the value at time t of the cash flow at time $t + \delta t$ of amount

$$f(\gamma, t + \delta t) - f(\gamma, t) \tag{13}$$

is zero.
Examples of futures' price processes are given by contracts with a set final future's price at a maturity T given by the value of an asset price $A(\gamma, t)$ at time T.

Martingale Relationship

Due to the Fundamental Theorem, there exists a pricing measure Q for which

$$E_{T-\delta t}[A(\gamma, T) - f(\gamma, t - \delta t)] = 0. \tag{14}$$

Hence

$$f(\gamma, t - \delta t)] = E_{T-\delta t}[A(\gamma, T)]. \tag{15}$$

We also have

$$f(\gamma, t - 2\delta t)] = E_{T-2\delta t}[f(\gamma, T - \delta t)] = E_{T-2\delta t}[A(\gamma, T)]. \tag{16}$$

By iterating this equation backward in time, we find that

$$F(\gamma, t) = E_t[A(\gamma, T)]. \tag{17}$$

Futures Contracts

Futures price processes are used in futures contracts. A futures contract is written with respect to a futures price process. Two parties, a long and a short party, and is stipulated in such a way that at all times after inception t, the short party in the contract pays the amount $F(\gamma, t) - F(\gamma, t - \delta t)$ to the long party of the contract. With this arrangement, the contract is always worth zero.

Forward Price Process

If $A(\gamma, t)$ is an asset price process and T a fixed maturity, the corresponding **forward price process** is defined as follows:

$$F(\gamma, t) = \frac{A(\gamma, t)}{Z_t(\gamma, T)}, \tag{18}$$

where

$$Z_t(\gamma, T) = E_t\left[\frac{g(\gamma, t)}{g(\gamma, T)}\right] \tag{19}$$

is the price process for a zero coupon bond, i.e. the asset paying one unit of account at maturity T.

Discretizing Calendar Time

In this section, we introduce a limit notion of arbitrage freedom in the continuous space and continuous time limit.

Let $h_n, n = 0, 1, ...$ be a sequence of lattice discretization mesh parameters and let us consider a lattice of the form $\Lambda_n = h_n \mathbb{Z}^D \cap (0, 1)$ where $D \geq 1$ is a dimension. We assume that $h_n \to 0$ as $n \to \infty$. We assume that also the corresponding time steps $\delta t_n \to 0$ as $n \to \infty$.

Discretizing Price Space

For each element n in the sequence of lattices, let $\mathcal{P}(\Lambda_n, k)$ be a path-space characterized by the incidence matrices $\kappa_n(\gamma, t_i)$ and let $g_n(\gamma, t_i)$ be a numeraire process. Let $A^s_n(\gamma, t_i)$ be a family of g-coherent non-anticipatory path functionals indexed by $s = 1, ...M$ with $M > 0$ on the time interval $t_i \in [t_0, t_j]$ where $j > 0$. Let Q_n is any probability measure with respect to which the processes A^s_n are g_n-discounted martingales. The existence of such a measure is assured by the Fundamental Theorem of Finance in the previous section.

Fundamental Theorem

Consider a situation with an asset price process given by the non-anticipatory functional $S_n(\gamma, t)$. Let us fix a strike price K and a maturity T. Let $F_n(\gamma, t; T)$ be the forward price process corresponding to $S_n(\gamma, t)$ and of maturity T.
Assume completeness and let $C_n(\gamma, t; K, T)$ be the price process for the call option contract with payoff $(S_n(\gamma, T) - K)_+$ at maturity T. According to the fundamental theorem, we have

$$C_n(\gamma, t; K, T) = E^{Q_n}\left[\frac{g(\gamma', t)}{g(\gamma', T)}(S_n(\gamma', T) - K)_+ \ |\gamma'_{t'} = \gamma_t \ \ \forall\ t' \leq t\right] \tag{20}$$

Assume that

$$\liminf_{n\to\infty} C_n(\gamma, t; K, T) > 0; \tag{21}$$

Continuous Paths

Let's specialize further to the case where possible paths are continuous in the sense that incidence matrices satisfy the following constraint:

$$\kappa_n(\gamma, t_i) = 0 \quad \textit{if} \quad |\gamma_{t_i} - \gamma_{t_{i+1}}| \geq 2; \tag{22}$$

Namely, a path can only hop from one value to the nearest neighbor at any given time.

The stop-loss-start-gain strategy

Assuming continuous paths, a call payoff can also be replicated by means of the trading strategy according to which whenever $F_n(\gamma, t; T) > K$ one holds a long position in the stock at time t and a short position $KZ_t(T)$ in a zero coupon bond maturing at time T .

In case $F_n(\gamma, t; T) <= K$ instead, according to this strategy one holds nothing. The strategy obviously replicates the call payoff and it is costly, i.e. it is not self-financing. The cost of the strategy is

$$E_t\Big[\sum_{t'=t+\delta t_n, \dots T} (S_n(\gamma, t') - S_n(\gamma, t' - \delta t))\delta(F_n(\gamma, t; T) > K)\delta(F_n(\gamma, t - \delta t; \quad (23)$$

Absence of Arbitrage?

Assuming absence of arbitrage, the cost to implement this strategy should match the call price as given in (20). As we shall see below, this consideration places severe limitations on the underlying process.

Theorem Assume that

- for all times $t' > t$ we have that

$$0 < \liminf_{n\to\infty} \frac{1}{h_n} E_t^{Q_n}\Big[\delta(F_n(\gamma, t; T) > K)\delta(F_n(\gamma, t-\delta t; T) \leq$$

$$< \limsup_{n\to\infty} \frac{1}{h_n} E_t^{Q_n}\Big[\delta(F_n(\gamma, t; T) > K)\delta(F_n(\gamma, t-\delta t; T) \leq K)$$

- for all possible paths γ and all times $t' > t$ we have that:

$$0 < \liminf_{n\to\infty} \frac{1}{h_n}(\bar{S}_n(\gamma, t') - \bar{S}_n(\gamma, t' - \delta t)) < \limsup_{n\to\infty} \frac{1}{h_n}(\bar{S}_n(\gamma, t') - \bar{S}_n(\gamma,$$

(25)

Proof

Due to the hypothesis above, the cost of the trading strategy is

$$E_t\Bigg[\sum_{t'=t+\delta t_n,\dots T} (S_n(\gamma, t') - S_n(\gamma, t'-\delta t))\delta(F_n(\gamma, t; T) > K)\delta(F_n(\gamma, t-\delta t; \quad (27)$$

However, due to the hypothesis of absence of arbitrage, this limit should converge in the limit as $n \to \infty$ to the call price $C_n(\gamma, t; K, T)$ which, by assumption, is finite and positive. This is not possible if the ratio $\frac{\delta t_n}{T h_n^2}$ can either be arbitrarily small or arbitrarily large in the limit as $n \to \infty$. Hence the conclusion.

Hurst Exponent

The condition in the Theorem can be rephrased in terms of the so called Hurst exponent H defined so that

$$E_t\left[|S_n(\gamma, t') - S_n(\gamma, t' - \delta t))|\right] = O\left(\delta t_n^H\right). \tag{28}$$

The conclusion is that the Hurst exponent needs to be equal to $1/2$ in order to achieve arbitrage freedom in a scalable fashion, i.e. also along a sequence $h_n \to 0$ as $n \to \infty$.

Diffusion Processes

In the next sections, we discuss Markovian diffusion processes with Hurst exponent 1/2 and show that they can indeed be taken as the basis for arbitrage free pricing models with continuous paths. We then discuss fractional Brownian motion and apply the theorem in this section to show that they allow for arbitrage trading strategies. Fractional Brownian motions are appealing empirically as they admit fat tailed and auto-correlated return distributions. However, lack of arbitrage freedom makes it impossible to use them for valuation theory. This raises a problem as empirically one observes fat tailed returns which indicate that historical asset price processes are not diffusions. In the concluding section in this chapter we discuss Markovian jump processes which admit fat tailed distribution. We also discuss stochastic volatility models which allow one to model auto-correlations albeit in a arbitrage-free way.