Financial Markets, Lecture 4

Claudio Albanese

King's College London, November 2nd, 2009

Farkas Lemma

In [Farkas, Julius (Gyula) (1902), "Über die Theorie der Einfachen Ungleichungen", Journal fr die Reine und Angewandte Mathematik 124: 127], Farkas established a result which became ubiquitous in Mathematical Economics and is known as **Farkas Lemma** or as the Fundamental Theorem of Linear Inequalities.
This result also lies at the basis of the Fundamental Theorem of Probability and Mathematical Finance in the next section. Here, we give a constructive proof of Farkas Lemma based on an explicit solution algorithm introduced by Avis-Kaluzni.

Farkas Lemma

Theorem Let **A** be a $n \times m$ matrix and let **b** be a real non-zero n-vector. Then either the primal system:

$$\mathbf{A}\mathbf{x} \le \mathbf{b}, \qquad \text{and} \qquad \mathbf{x} \ge 0 \tag{1}$$

has a solution or the dual system

$$\mathbf{A}^T\mathbf{y} \ge 0, \qquad \text{and} \qquad y \cdot \mathbf{b} < 0 \tag{2}$$

has a solution but never both.

Proof of Farkas Lemma

We begin by noticing that there cannot exist both a vector **x** and a vector **y** satisfying the conditions of the theorem. For otherwise, $0 > y^T b \geq y^T Ax \geq 0$. What remains to be shown is that in case the primal system does not admit a solution, then the dual system does.

The proof is based on a iterative algorithm introduced by Avis-Kaluzni that solves the system:

$$\mathbf{Ax} \leq \mathbf{b}, \qquad \text{and} \qquad x \geq 0. \tag{3}$$

The algorithm is designed in such a way to halt after a finite number of iterations and to either give rise to a solution or to allow one to conclude that a solution does not exist and that instead there is a solution for the dual problem.

Proof of Farkas Lemma

It is useful to explain the algorithm by means of examples. Consider the linear system of inequalities

$$\begin{cases} -x_1 - 2x_2 + x_3 & \leq & -1 \\ x_1 - 3x_2 - x_3 & \leq & 2 \\ -x_1 - 2x_2 + 2x_3 & \leq & -2 \\ x_1, x_2, x_3 & \geq & 0. \end{cases} \tag{4}$$

Proof of Farkas Lemma

This system can be turned into an equivalent linear system of equalities by introducing three additional variables x_4, x_5, x_6 called *slacks*:

$$\begin{cases} x_4 &= -1 + x_1 + 2x_2 - x_3 \\ x_5 &= 2 - x_1 + 3x_2 + x_3 \\ x_6 &= -2 + x_1 + 2x_2 - 2x_3. \end{cases} \tag{5}$$

with the positivity constraint $x_i \geq 0$ for $i = 1, ..6$.

Proof of Farkas Lemma

This linear system provides an example of a **dictionary**, i.e. a system of linear equations where

- variables are constrained to be non-negative,
- variables are divided into left-hand-side (l.h.s.) variables and right-hand-side (r.h.s.) variables,
- each l.h.s. variable appears only in one equation and is on the left hand side with a coefficient 1,
- each r.h.s. appears only on the right hand side of the linear equations and may appear on any equation with any coefficient,
- known numeric terms appear on the right hand side of the equations.

Proof of Farkas Lemma

Dictionaries can be transformed into each other by means of **pivoting operations** whereby

- one selects a r.h.s. variable,
- one uses one particular linear equation to express it in terms of the r.h.s. variables and the l.h.s. variable appearing in that equation,
- one inserts this expression into the other equations, thus obtaining a new dictionary whereby the selected r.h.s. variable becomes a l.h.s. variable, while the l.h.s. variable in the selected equation becomes a r.h.s. variable.

Proof of Farkas Lemma

The solution algorithm consists in an iteration of pivoting transformations aimed at obtaining a dictionary whereby the numeric terms on the right hand side are all non-negative. If this turns out to be possible, then a solution is found by setting all r.h.s. variables to zero.

Proof of Farkas Lemma

To continue with the example above, we find the smallest-indexed l.h.s. variable with a positive coefficient appearing in an equation with a negative known term on the right hand side. In this case, we are looking at x_4. In the equation for x_4, let us pivot around the r.h.s. variable with the smallest index that appears with a positive coefficient (in this case x_1. We find

$$\begin{cases} x_1 & = & 1 - 2x_2 + x_3 + x_4 \\ x_5 & = & 1 + 5x_2 - x_4 \\ x_6 & = & -1 - x_3 + x_4 . \end{cases} \tag{6}$$

Proof of Farkas Lemma

Now there is only one equation left with a negative known term on the right hand side. We repeat the procedure and choose the smallest indexed right hand variable appearing with a positive coefficient in this equation, namely x_4 and carry out a pivot operation around it. We find the new dictionary

$$\begin{cases} x_1 &= 2 - 2x_2 + 2x_3 + x_6 \\ x_4 &= 1 + x_3 + x_6 \\ x_5 &= 0 + 5x_2 - x_3 - x_6. \end{cases} \tag{7}$$

Since the known terms on the right hand side are all positive, we conclude that a solution is given by $x_1 = 2, x_2 = 0, x_3 = 0$.

Proof of Farkas Lemma

Although in this example we arrived to a solution, in general the algorithm may halt otherwise. This happens when there are equations with a negative known term and in these equations all right hand variables appear with negative coefficients. If this happens, then the above rules do not apply and the algorithm halts. In this case, one can show that a solution does not exist.

Proof of Farkas Lemma

To explain, consider a second example

$$\begin{cases} x_4 &= \quad 3 + x_1 - 2x_2 - x_3 \\ x_5 &= -17 - 3x_1 + 2x_2 - x_3 \\ x_6 &= 19 + x_1 + 6x_2 + 23x_3. \end{cases} \tag{8}$$

with the positivity constraint $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$.

Proof of Farkas Lemma

The algorithm proceeds as before by choosing the equation for x_5 and solving for x_2:

$$\begin{cases} x_2 & = & \frac{17}{2} + \frac{3}{2}x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_5 \\ x_4 & = & -14 - 2x_1 - 2x_3 - x_5 \\ x_6 & = & 70 + 10x_1 + 26x_3 + 3x_5. \end{cases} \tag{9}$$

Proof of Farkas Lemma

At this point the algorithm halts as in the second equation all right hand variables appear with a negative sign and the known term is also negative. At closer inspection one realizes that this equation does not possibly have a non-negative solution. Since the problem is equivalent to the original one, that one also does not admit a solution.

Proof of Farkas Lemma

Let's now discuss the general case of a system with m equations and n unknowns. We introduce m slack variables and re-write the system in dictionary form:

$$x_{l(i)} = d_{l(i)} + \sum_{j=1}^{n} C_{ij} x_{r(j)}. \tag{10}$$

Here, $x_{r(j)}, j = 1, ...n$ is the collection of right hand variables and $x_{l(i)}, i = 1, ...m$ is the collection of left hand variables. We then carry out the algorithm: if all the known terms $d_{l(i)}$ are non-negative then the algorithm halts and a solution is found by setting all the r.h.s variables to zero.

Proof of Farkas Lemma

Otherwise, from among the equations with a negative $d_{l(i)}$ select the one with the least index $l(i)$. In this equation, check if there are r.h.s. variables which appear with a positive coefficient. If there aren't such variables the algorithm halts and the conclusion is that no solution exists. Otherwise, from among the r.h.s. variables with positive coefficients, select the one with the least index $r(j)$ and pivot around it. Then iterate.

Proof of Farkas Lemma

If the variable x_{n+m} moves from the right to the left during the cycle, based on the previous rules it ought to do so as a consequence of pivoting about it in an equation of the form:

$$x_{l(i)} = d_{l(i)} + \sum_{j=1}^{n-1} C_{ij} x_{r(j)} + C_{i,n} x_{n+m}. \tag{11}$$

We need to have $d_{l(i)} < 0$ or else the equation would not be selected. We also have $C_{i,n} > 0$ or else we would not consider the variable x_{n+m} as a candidate to pivot around. Finally, we need to have $C_{ij} \leq 0$ for all $j = 1, ..n - 1$ or else we would pick a variable of lesser index to pivot around. This equation shows that, if $x_i \geq 0$ for all $i = 1, ...n + m - 1$, then $x_{n+m} > 0$.

Proof of Farkas Lemma

If the variable x_{n+m} moves from the left to the right instead, it does so while pivoting about it in an equation of the form:

$$x_{n+m} = d_{n+m} + \sum_{j=1}^{n-1} C_{ij} x_{r(j)}. \tag{12}$$

We need to have $d_{n+m} < 0$ or else the equation would not be selected. We also need to have $d_{l(i)} \geq 0$ for all indices i such that $l(i) \neq n + m$ or another equation would be selected.
We conclude that, there is a solution of the system of linear inequalities for which $x_i \geq 0$ for all $i = 1, ...n + m - 1$ and $x_{n+m} < 0$. To obtain this solution one just has to set to zero all right hand variables.

Proof of Farkas Lemma

Since this eventuality contradicts the conclusion we reached by assuming that x_{n+m} moves from the right to the left during the algorithm, we arrive to the conclusion that just one of the following three cases must hold:

- The variable x_{n+m} never moves from one side to the other of the equation during the course of the algorithm;
- The variable x_{n+m} moves from the left to the right just once and then never moves again to the left;
- The variable x_{n+m} moves from the right to the left just once and then never moves again to the right.

Proof of Farkas Lemma

Since only a finite number of combinations of left and right hand variables is possible, either the algorithm stops after a finite number of steps or it develops a cycle. We can show that the latter case is not possible by contradiction.
As we have seen, it is not possible that over the cycle the variable x_{n+m} moves from the left to the right and then from the right to the left. One possibility is that the variable x_{n+m} sits on the right hand side throughout the cycle; in this case, one can set this variable to zero and pass to a system with one fewer variable and the same number of equations which would repeat the same cycle.

Proof of Farkas Lemma

The alternative possibility is that x_{n+m} sits constantly on the left hand side, then one can eliminate the corresponding equation and still obtain a cycling dictionary. Either way, we reduce to a system with either one equation fewer or one variable less. The inductive argument can be repeated until one either eliminates all equations or one eliminates all right hand variables. Either way, such a system cannot cycle. So we conclude that our algorithm halts in a finite number of steps. In the worst case scenario, the algorithm halts after having explored all possible rearrangements of left and right and variables, i.e. after at most $\begin{pmatrix} n \\ n+m \end{pmatrix}$ steps.

Proof of Farkas Lemma

Cycles being impossible, there are two ways the algorithm can halt. Either it finds a solution. In this case the dual system has no solution.

Proof of Farkas Lemma

A second possibility is that one arrives to an incosistent equation of the form

$$x_{l(i)} = d_{l(i)} + \sum_{j=1}^{n} C_{ij} x_{r(j)}. \tag{13}$$

with $d_{l(i)} < 0$ and $C_{ij} \leq 0$ for all j. In this case, we define the vector y_k, $k = 1, ...m$ so that

- $y_k = C_{ij}$ if there is a j for which $r(j) = n + k$,
- $y_k = -1$ if there is a j for which $l(i) = n + k$,
- $y_k = 0$ otherwise.

Proof of Farkas Lemma

The starting system has the form

$$x_{n+k} = b_{n+k} - \sum_{j=1}^{n} A_{kj} x_j. \tag{14}$$

The incosistent equation can be obtained by multiplying each equation in this original system by y_k and summing over k, i.e. it is equivalent to the equation

$$\sum_{k=1}^{m} x_{n+k} y_k = \sum_{k=1}^{m} y_k b_{n+k} - \sum_{k=1}^{m} \sum_{j=1}^{n} y_k A_{kj} x_j. \tag{15}$$

Comparing terms, we find that

$$\sum_{k=1}^{m} y_k b_{n+k} = d(l(i)) < 0. \tag{16}$$

Proof of Farkas Lemma

Moreover

- $-\sum_{k=1}^{m}\sum_{j=1}^{n} y_k A_{k,j} = C_{kl} <= 0$, for each l such that $r(l) = j$
- $-\sum_{k=1}^{m}\sum_{j=1}^{n} y_k A_{k,j} = -1 < 0$, if there is an l such that $l(i) = j$,
- $-\sum_{k=1}^{m}\sum_{j=1}^{n} y_k A_{k,j} = 0$, otherwise.

In conclusion, the vector **y** solves the dual problem

$$\mathbf{y}\cdot\mathbf{b} < 0, \quad \text{and} \quad A^T\mathbf{y} \geq 0. \tag{17}$$

Fundamental Theorem of Finance

Consider a family of time points $t_i = t_0 + i\delta t$ where t_0 is today's date, δt is a constant step and $i = 0, 1, 2, ...$ is an integer. Consider also a discrete state space $\Lambda = \{0, ...d - 1\}$ where $d \geq 1$. Let $\mathcal{P}(\Lambda)$ denote the set of all paths $\gamma = (\gamma_i)_{i=0,1,...}$ with $\gamma_i \in \Lambda$. γ_i is the state variable of path γ at time t_i.

Fundamental Theorem of Finance

Definition. If $j \geq 0$ is a non-negative integer, a function $\phi(\gamma, t_i)$ with $\gamma \in \mathcal{P}(\Lambda), i = 0, 1, ...$ is called *step-j non-anticipatory* if

$$\phi(\gamma, t_i) = \phi(\gamma', t_i) \tag{18}$$

for all i and whenever $\gamma_k = \gamma'_k$ for all $k \leq i + j$.

Fundamental Theorem of Finance

Definition. A pathspace $\mathcal{P}(\Lambda, \kappa)$ is characterized by a sequence of *incidence matrices* taking only values 0 and 1 and given by step-1 non-anticipatory functions $\kappa(\gamma, t_i) \in \{0, 1\}$ such that if $\kappa(\gamma, t_i) = 0$ for some $i \geq 0$ then $\kappa(\gamma, t_k) = 0$ for all $k \geq i$. A path γ belongs to the set $\mathcal{P}(\Lambda, k)$ if $\kappa(\gamma, t_i) = 1$ for all $i \geq 0$.

Numeraire Process

Definition. A real valued process adapted to the pathspace $\mathcal{P}(\Lambda, \kappa)$ is given by a real valued step-0 non-anticipatory function $A(\gamma, t_i)$.
Pricing is carried out relative to a valuation benchmark, also called numeraire.
Definition. A numeraire is a positive valued adapted process $g(\gamma, t_i) > 0$.

Numeraire Process

An example of a numeraire is given by the price of a commodity with negligible carry costs and negligible convenience yield such as, for instance, gold. A second example of a numeraire is defined through a positive valued process $r(\gamma, t_i)$ interpreted as a short rate, i.e. the money market account process given by

$$B(\gamma, t_i) = (1 + \delta t r(\gamma_0, t_0)) \quad \dots \quad (1 + \delta t r(\gamma_{i-1}, t_{i-1})). \qquad (19)$$

Arbitrage Freedom

Definition. Let $\mathcal{P}(\Lambda, k)$ be a pathspace characterized by the incidence matrices $\kappa(\gamma, t_i)$ and let $g(\gamma, t_i)$ be a numeraire process. Let $A^s(\gamma, t_i)$ be a family of non-anticipatory path functionals indexed by $s = 1, ...M$ with $M > 0$ on the time interval $t_i \in [t_0, t_j]$ where $j > 0$. The family of processes $A^s(\gamma, t_i)$ is called g-coherent if for any $t_i \in [t_0, t_j]$ and any set of coefficients $\zeta^s, s = 1, ...M$, the following property holds: if γ is a possible path for which $\kappa(\gamma, t_i) = 1$ and

$$\frac{1}{g^s(\gamma, t_{i+1})} \sum_s \zeta^s A^s(\gamma, t_{i+1}) - \frac{1}{g^s(\gamma, t_i)} \sum_s \zeta^s A^s(\gamma, t_i) > 0 \quad (20)$$

then there is a second possible path γ' such that $\gamma'_k = \gamma_k$ for all $k \leq i$ and $\kappa(\gamma', t_i) = 1$ and

$$\frac{1}{g^s(\gamma\prime, t_{i+1})} \sum_s \zeta^s A^s(\gamma\prime, t_{i+1}) - \frac{1}{g^s(\gamma, t_i)} \sum_s \zeta^s A^s(\gamma, t_i) < 0. \quad (21)$$

Transition probability kernels

Definition. An elementary transition probability kernel on $\mathcal{P}(\Lambda, k)$ is a family of 1-step non-anticipatory path functionals $q(\gamma, t_i)$ defined for $i = 0, 1, .., \gamma \in \mathcal{P}(\Lambda, k)$ satisfying the following properties:

(i) $q(\gamma, t_i) \geq 0$,

(ii) $\sum_{\gamma' : \gamma'_k = \gamma_k, k \leq i} q(\gamma', t_i) = 1$,

(iii) $q(\gamma, t_i) > 0$ if and only if $\kappa(\gamma, t_i) = 1$, while otherwise $q(\gamma, t_i) = 0$.

Expectations

Let us fix a time $t_j > t_0$ and let $A(\gamma, t_i), i = 0, ...j$ be a process. The expectation of this process at time t_i prior to t_j with respect to the kernel $q(\gamma, t_i)$ is a process denoted by $E^q_{t_i}[A(\gamma, t_j) \mid \{\gamma_k\}_{k\leq i}]$ and defined as follows:

$$E^q_{t_i}[A(\gamma, t_j)|\{\gamma_k\}_{k\leq i}] = \sum_{\gamma' : \gamma'_k = \gamma_k \forall k \leq i} q(\gamma', t_i) \; .... \; q(\gamma'_, t_{j-1}) A(\gamma', t_j). \tag{22}$$

Conditional Probabilities

This formula can be recast in terms of conditional probabilities of a path given by the functional

$$p(\gamma, t_i, t_j) \equiv q(\gamma, t_i) \;....\; q(\gamma, t_{j-1}). \tag{23}$$

In terms of this functional, we can express expectations in (22) as follows:

$$E^q_{t_i}[A(\gamma, t_j) | \{\gamma_k\}_{k \le i}] = \sum_{\gamma' : \gamma'_k = \gamma_k \forall k \le i} p(\gamma', t_i, t_j) A(\gamma', t_j). \tag{24}$$

Conditional Probabilities

The interpretation of $p(\gamma, t_i, t_j)$ is that this is the probability for a path to be equal to γ in the time interval $[t_{i+1}, t_j]$ conditioned to knowing that it coincides with γ on the preceding time interval $[t_0, t_i]$.

Martingales

Definition. Let $g(\gamma, t_i)$ be a numeraire asset, the adapted process $A(\gamma, t_i)$ is called g-discounted martingale with respect to the elementary kernel $q(\gamma, t_i)$ if

$$A(\gamma, t_i) = E^q_{t_i}\left[\frac{g(\gamma, t_i)}{g(\gamma, t_j)} A(\gamma, t_j) \middle| \{\gamma_k\}_{k \le i}\right]. \tag{25}$$

where E^q denotes the expectation with respect to the elementary transition probability kernels $q(\gamma, t_i)$.

Martingales

As an example, consider the case where the numeraire is given by the money market account $B(\gamma, t_i)$ in (19). Fix a time $t_j > t_0$ and let $A(\gamma, t_j)$ be a process. The discounted expectation of this process at time t_i prior to t_j with respect to the money market account $B(\gamma, t_i)$ can be expressed as follows:

$$E^q_{t_i}\left[\frac{B(\gamma,t_i)}{B(\gamma,t_j)}A(\gamma, t_j) \,\middle|\, \{\gamma_k\}_{k\le i}\right]$$

$$= \sum_{\gamma':\gamma'_k=\gamma_k\ \forall k\le i} q(\gamma', t_i) \quad .... \quad q(\gamma', t_{j-1}) \, \frac{B(\gamma',t_i)}{B(\gamma',t_j)} A(\gamma', t_j). \tag{26}$$

Martingales

Elementary discounted transition probability kernels defined as

$$q^D(\gamma, t_i) = \frac{1}{1 + \delta tr\left(\gamma, t_i\right)} q(\gamma, t_i) \tag{27}$$

can be used to implicitly account for the numeraire asset in the path expansion for discounted expectations by recasting it as follows:

$$\begin{aligned} & E^q_{t_i}\left[\tfrac{B(\gamma,t_i)}{B(\gamma,t_j)} A(\gamma, t_j) \middle| \{\gamma_k\}_{k\le i} \right] \\ &= \textstyle\sum_{\gamma' : \gamma'_k = \gamma_k \ \forall k \le i} q^D\left(\gamma', t_i\right) \quad \dots. \quad q^D\left(\gamma', t_{j-1}\right) \, A(\gamma', t_j). \end{aligned} \tag{28}$$

Martingales

The family of adapted processes $A^s(\gamma, t_i), s = 1, ..M$ is called a family of g-discounted equivalent martingales if there exists an elementary kernel $q(\gamma, t_i)$ on the path-space $\mathcal{P}(\Lambda, k)$ with respect to which the processes $A^s(\gamma, t_i)$, $s = 1, ..M$, are all g-discounted martingales.

Fundamental Theorem of Finance

(First Fundamental Theorem of Finance) Let $g(\gamma, t_i)$ be a numeraire process and let $A^s(\gamma, t_i), s = 1, ...M$ be family of processes adapted to $\mathcal{P}(\Lambda, k)$ and defined on the time interval $t_i \in [t_0, t_j]$. Then $A^s, s = 1, ...M$ is a family of equivalent g-discounted martingales if and only if they are a g-coherent family.

Proof of the Fundamental Theorem of Finance

This theorem was first stated and proved by Bruno de Finetti in [**?**]. The proof depends on Farkas Lemma in the previous section and goes as follows.
Assume that $A^s(\gamma, t_i)$ is a g-coherent family of processes. Let $i \geq 0,\ t_i \in [t_0, t_j]$. We need to show that for all $i \geq 0$, there exists a transition probability kernel $q(\gamma, t_i)$ such that

$$q(\gamma, t_i) > 0 \text{ if and only if } \; k(\gamma, t_i) = 1, \tag{29}$$

with respect to which the processes $A^s(\gamma, t_i)$ are discounted martingales

Proof of the Fundamental Theorem of Finance

In other words, we need to show that

$$\sum_{\gamma':\gamma'_k=\gamma_k\ \forall k\leq i} q(\gamma',t_i)A^s(\gamma',t_{i+1})\frac{g(\gamma',t_i)}{g(\gamma',t_{i+1})} = A^s(\gamma,t_i) \qquad (30)$$

for all $s = 1, ...M$, all $i = 0, ..j$ and all $\gamma \in \mathcal{P}(\Lambda, k)$. This is sufficient as, by iterating this equation one arrives to the discounted martingale condition over arbitrarily long time intervals.

Proof of the Fundamental Theorem of Finance

It is convenient to recast this last equation (30) as follows:

$$\sum_{\gamma':\gamma'_k=\gamma_k\ \forall k\leq i} q(\gamma',t_i)\left(A^s(\gamma',t_{i+1})\frac{g(\gamma',t_i)}{g(\gamma',t_{i+1})}-A^s(\gamma,t_i)\right)=0. \tag{31}$$

Proof of the Fundamental Theorem of Finance

Let γ^abe a family of paths where $a = 1, \ldots, n$ with the following properties:

(i) they coincide for $t \leq t_i$, i.e. $\gamma^a_k = \gamma^b_k$ for all $a, b = 1, \ldots, n$ and all $k \leq i$;

(ii) they differ at time t_{i+1}, i.e. $\gamma^a_{i+1} \neq \gamma^b_{i+1}$ if $a \neq b$.

(iii) For all $a = 1, \ldots, n$ we have that $k(\gamma, t_i) = 1$;

(iv) If y satisfies $k(\gamma, t_i) = 1$ then there is a path γ^a in the family such that $y = \gamma^a_{i+1}$.

Proof of the Fundamental Theorem of Finance

Let

$$w^{as} = A^s(\gamma^a, t_{i+1}) \frac{g(\gamma^a, t_i)}{g(\gamma^a, t_{i+1})} - A^s(\gamma^a, t_i). \tag{32}$$

Let $\mathcal{V}$ be a n-dimensional vector space with basis vectors $\mathbf{v}^1, ..., \mathbf{v}^n$. Let $\mathbf{w}^s$ be the vector in $\mathcal{V}$ of components

$$\mathbf{w}^s = \sum_{a=1}^{n} w^{as} \mathbf{v}^a. \tag{33}$$

Proof of the Fundamental Theorem of Finance

It suffices to show that there is a vector $\mathbf{q} = (q^a)$ such that $q^a \geq 0$, $\forall a = 1, \ldots, n$, $\sum_{a=1}^{n} q^a = 1$ and

$$\mathbf{q} \cdot \mathbf{w}^s \equiv \sum_{a=1}^{n} q^a w^{as} = 0 \quad \forall s = 1, \ldots, M. \tag{34}$$

The coherence hypothesis can be recast in this language as the statement according to which a vector (ζ^s) satisfies

$$\sum_{s=1}^{M} \zeta^s w^{as} \geq 0, \quad \forall a = 1, ..., n \tag{35}$$

only if it is the zero vector, i.e. $\zeta^s = 0$.

Proof of the Fundamental Theorem of Finance

The system

$$\begin{cases} \sum_{a=1}^{n} q^a = 1 \\ \sum_{a=1}^{n} q^a w^{as} = 0 \\ q^a \geq 0, \end{cases} \tag{36}$$

can be recast as a primal system of linear inequalities in the standard form $Aq = c$, $q \geq 0$, where

$$\mathbf{A} = \begin{pmatrix} 1 \ \ 1 \ \ \ldots. \ 1 \\ \\ W^T \end{pmatrix} \ \text{and} \ \ c = \begin{pmatrix} 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix} \tag{37}$$

Proof of the Fundamental Theorem of Finance

The corresponding dual system is

$$\xi_{-1} + \sum_{s=1}^{M} w^{as} \xi^s \geq 0, \quad \xi_{-1} < 0, \tag{38}$$

i.e. $\sum_{s=1}^{M} w^{as} \xi^s > 0$. Assuming coherence, the dual system does not have a solution. Hence the primal equation does and this establishes one direction of the theorem. Vice versa, thanks again to Farkas Lemma, if the dual has a solution, i.e. there is no g-coherence, then the processes are not g-discounted martingales.