

Financial Markets, Lecture 2

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The Stock Price Process

Consider a discrete and finite state space Λ , a time coordinate $t_i = i\delta t$. Consider also two asset price processes: that for a stock price given by a non-anticipatory functional $S(\gamma, t)$ and that for a money market account given by a process $M(t)$ that we assume here depends only on time but not on the path γ .

In other words, we assume in this section that interest rates do not depend on any events except for calendar time. Finally, we assume that the two assets price processes above satisfy the principle of no arbitrage.

Forward Contracts

A **forward contract** is characterized by a fixing date t_0 , a maturity date $T > t_0$ and a **forward price** F . The contract states that one party will pay to the other party an amount equal to $S(\gamma, T) - F$ at time T , where γ is the path realized up to time T .

Theorem. Assuming no arbitrage, if the forward price is equal to

$$F = S_0 \frac{M(T)}{M(t_0)}, \quad (1)$$

then the value of the forward contract at time t_0 is equal to zero.

Forward Contracts

Suppose that the value of the forward contract is negative. Then one can form a portfolio at time t_0 with a long position in one forward contract, a short position in a share of the stock and a long position in cash deposited in an interest bearing account. At time t_0 , one would be paid to form such a position. However at time T the position would always be worth zero. Hence this trade would represent an arbitrage, contrary to the hypothesis of absence of arbitrage.

Similarly, if the value of the forward contract was positive one can devise a trade similar to the one above except that positions would have precisely the opposite sign, such that an arbitrage would result.

Call and Put Options

A **call option contract** is characterized by a fixing date t_0 , a maturity date $T > t_0$ and a **strike price** K . The contract states that one party will pay to the other party an amount equal to $(S(\gamma, T) - K)_+$ at time T , where γ is the path realized up to time T and $(a)_+ = a$ in case $a > 0$ while it is zero otherwise.

A **put option contract** is characterized by a similar set of parameters as a call option except that the payoff is $(K - S(\gamma, T))_+$.

Put-Call Parity

Theorem. Assuming no arbitrage, if $C(\gamma, t; K, T)$ and $P(\gamma, t; K, T)$ are the price processes for a call and a put option of strike K and maturity T and if $F(\gamma, t; K, T)$ is the price process for a forward contract with forward price K and maturity T , then we have that

$$C(\gamma, t; K, T) - P(\gamma, t; K, T) = F(\gamma, t; K, T). \quad (2)$$

Put-Call Parity

The proof of the put-call parity relationship descends from the principle of no arbitrage once one recognizes that the payoff at time T of a combination of a long position in a call and a short position in a put of the same strike K is precisely equal to the payoff of a forward contract with forward price also equal to K .

A Double Barrier Option

In this subsection, we assume that the money market account is a process identically equal to one, i.e. $M(t) = 1$. This amounts to assuming that interest rates are zero at all times.

A second assumption we make is that the stock price values that the stock price $S(\gamma, t)$ can assume are restricted to the set $j\delta S$ where $j = 0, 1, \dots$ and δS is fixed (equal for instance to one cent). Furthermore, we make the following **continuity assumption**:

$$|S(\gamma, t + \delta t) - S(\gamma)| \leq \delta S. \quad (3)$$

In other words, in any given time interval the stock price can either go up by δS , down by the same amount or can stay at the same level.

A Double Barrier Option

Let $S_0 = S(\gamma, t_0)$ be the value of the stock price at the fixing date t_0 , also called the **spot price**. Consider a perpetual barrier option with barriers at $L < S_0$ and $R > S_0$. We assume that both the level L and the level U are integer multiples of δS .

The contract stipulates that the holder receives an amount R_U if the upper barrier is passed before the lower barrier and an amount R_L if the lower barrier is passed before the upper barrier. Since the option is perpetual, the maturity is infinite and expiration occurs only when one of the barriers is hit.

Pricing and Hedging

The price and hedging strategy for a perpetual double barrier under the assumptions above can be determined by noticing that the option can be replicated by holding a position in cash and in the underlying. The price of such a portfolio is linear in the stock, i.e. has the form

$$P_t = a + bS_t. \quad (4)$$

If the position parameters a and b are determined in such a way that

$$\begin{cases} R_U = a + bU, \\ R_L = a + bL, \end{cases} \quad (5)$$

then the hedge can be unwound precisely at the hitting time to replicate the desired payoff.

Pricing and Hedging

The solution of the linear system in equation (5) is

$$\begin{cases} a = \frac{R_L U - R_U L}{U - L}, \\ b = \frac{R_U - R_L}{U - L}, \end{cases} \quad (6)$$

Hence the price is

$$P_0 = a + bS_0 = \frac{R_L U - R_U L}{U - L} + \frac{R_U - R_L}{U - L} S_0 \quad (7)$$

Variance Swaps

A variance swap is a contract written on the realized variance of a stock or exchange rate over a period of time. More precisely, consider the path of a stock price $S(\gamma, t)$ over the time period $[0, T]$ and suppose the stock price is observed every day at time $t = t_0 + i\delta t, i = 0, 1, \dots, N$. I am denoting with N the number of days to the maturity date T .

The realized variance of arithmetic returns is defined as follows:

$$RV = \frac{1}{N} \sum_{i=0}^{N-1} \left(\frac{S(\gamma, t_{i+1}) - S(\gamma, t_i)}{S(\gamma, t_i)} \right)^2 - SR^2. \quad (8)$$

Here, SR is called swap rate.

Equilibrium Swap Rates

The problem is to (i) determine the equilibrium swap rate SR such that this payoff has zero value at contract inception and (ii) find a hedging strategy involving trading in the stock and European options.

Continuity Assumptions

For simplicity's sake, we retain the two assumptions made in the previous section, i.e. that interest rates are zero and that stock prices either stay constant or hop by an amount δS at each elementary time step.

Furthermore, we assume that returns are so small that the Taylor expansion for a log-return can be truncated to the second order, or more precisely that:

$$\log \frac{S_{t_{i+1}}}{S_{t_i}} = \log \left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} + 1 \right) \approx \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} - \frac{1}{2} \left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \right)^2. \quad (9)$$

Here, to simplify notations, we denote $S(\gamma, t) = S_t$. We also use the Taylor expansion for the logarithm according to which

$$\log(1 + x) = x - \frac{1}{2}x^2 + O(x^3) \quad (10)$$

Hedging Strategy

Based on the approximation above, let us consider the following sum:

$$\sum_{i=0}^{N-1} \log \frac{S_{t_{i+1}}}{S_{t_i}} \approx \sum_{i=0}^{N-1} \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} - \frac{1}{2} \sum_{i=0}^{N-1} \left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \right)^2. \quad (11)$$

Hedging Strategy

By rearranging, we can re-express realized variance as follows:

$$\frac{1}{N} \sum_{i=0}^{N-1} \left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \right)^2 \approx -\frac{2}{N} \sum_{i=0}^{N-1} \log \frac{S_{t_{i+1}}}{S_{t_i}} + \frac{2}{N} \sum_{i=0}^{N-1} \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}. \quad (12)$$

The right hand side of this equation can be interpreted financially as the combination of a trading strategy in the stock and European options. In fact

$$\frac{2}{N} \sum_{i=0}^{N-1} \log \frac{S_{t_{i+1}}}{S_{t_i}} = \frac{2}{N} \log \left(\frac{S_T}{S_{t_0}} \right). \quad (13)$$

Hedging Strategy

Finally, consider the sum

$$\frac{2}{N} \sum_{i=0}^{N-1} \frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}. \quad (14)$$

If interest rates are zero as assumed, then the one-day forward price at time t_i is S_{t_i} . Hence this sum can be obtained by taking a position in $\frac{2}{NS_{t_i}}$ forward contracts of maturity one day on each day i . Since these contracts are at equilibrium, this strategy is self-financing and can be implemented at zero cost.

Equilibrium Swap Rates

From this calculation we can draw a conclusion about the equilibrium swap rate SR defined as the rate of a variance swap for which the value of the contract is zero. Namely,

Theorem. The equilibrium swap rate is approximately equal to the square root of the price of the European option of payoff

$$-\frac{2}{N} \log \left(\frac{S_T}{S_{t_0}} \right). \quad (15)$$

Replication of logarithmic payoffs

To replicate a log payoff approximately, one may use call and put options as indicated in the Figure below.

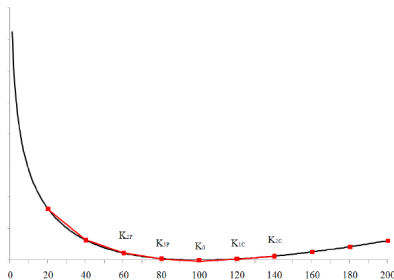


Figure: Replication of a logarithmic payoff