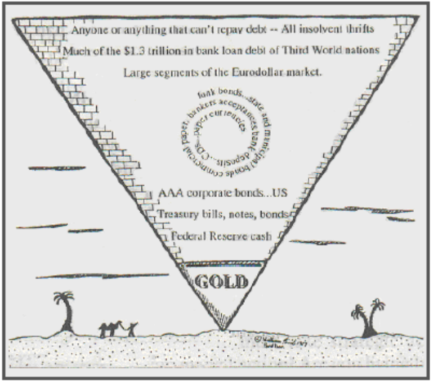
Financial Markets, Lecture 1

Claudio Albanese

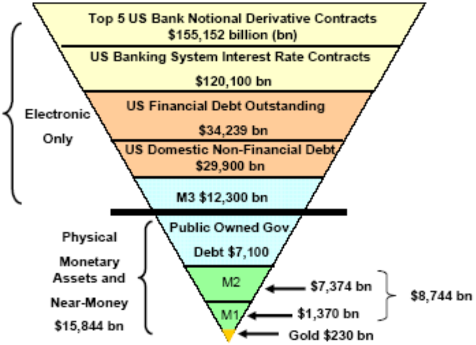
King's College London, October 4th, 2009

The structure of the monetary system as it was 50 years ago was described by John Exeter, a former Fed Governor, as an inverted pyramid with gold at the tip, deposit accounts, government debt, equity and third world debt at the higher layers:



The Money Pyramid

The money pyramid grew over the years to accommodate layers upon layers of increasingly complex derivative contracts:



Economic Rationale and Risks

- The economic rationale of the money pyramid is to ease trading of illiquid assets by establishing bridges to liquid ones.
- The money pyramid is held together by mathematical models for derivative valuation.
- Valuation models should reflect the real economy;
- The real economy certainly reflects valuation models.
- If valuation models fail to be economically realistic, dislocations occur and the economy fails.

Derivative Valuation

- Financial contracts are worded according to the principles of temporal modal logic (i.e. predicate logic with a notion of time and of possible future events).
- The basic valuation principle is the ancient principle of **no-arbitrage**: if a trading strategy can possibly give rise to a profit then it can also possibly give rise to a loss.

Pricing Theory

- The linchpin of pricing theory is given by the Fundamental Theorem of Finance, a result first established back in the 1931 by Bruno de Finetti who placed it at the foundations of subjectivist Probability Theory.
- Loosely speaking, the Fundamental Theorem asserts that there is no arbitrage if and only if all asset prices can be represented mathematically as discounted expectations of future payoffs starting from a single probability distribution over possible future scenarios.
- To be covered in the second part of the course.

Temporal Modal Logic

- Aristotle (384 BC 322 BC): Predicate logic, syllogisms
- Aristotle Modal Logic: Possible vs. necessary
- Ibn Sina (Avicella) (980-1037 C.E.): Temporal modal logic
- Bruno de Finetti (1932): Logic foundations of probability theory
- Jean Marc Eber (2003): Lexical analyzers for contract payoff language

Financial Events

Financial events occur whenever two parties transact by exchanging assets or contractual obligations. The information regarding the transaction is often stored and comes into existence. Sometimes the information is reported to some select market participants and in some cases it is broadcasted to the general public.
For the purpose of representing financial events, time can always be regarded as discretely metered. Typically the discretization involves regular time steps except for pauses when markets close overnight or during business holidays.
The set of times when data is generated is represented by an increasing sequence of dates $t_i, i = 0, ...n_i$, whereby $t_0 = 0$ is a reference date.

Finiteness

Although the universe of all Financial events occurring at any given point in time is huge and of unwieldy complexity, it is safe to say that it is discrete and finite. In fact, stock prices, foreign exchange rates and commodity prices are quoted as multiples of a cent or an elementary discrete amount. Interest rates are quoted as multiples of a basis point or 0.01%. There are also natural upper and lower bounds to the values that financial data can take. For instance, exchange traded asset prices are non-negative and they cannot be arbitrarily large.

The universe of all values that financial data can possibly take at any given time t_i is represented by a finite set Ω_i. An element $\omega \in \Omega_i$ is called an **event at time** t_i.

Data Formatting

The first step in building a valuation or risk model involves filtering out the set of all available data by taking a cross-section. Financial contracts provide a guideline for the procedure. Contracts are expressed in terms of natural language which, in order to be meaningful and legally enforceable, is necessarily bound by rigorous logic constraints and can thus be mapped into consistent mathematical structures. Contract logic itself provides a first filter. If one is considering the universe of all contracts referring exclusively on JPY LIBOR rates, data referring to commodities or other currencies is not relevant.

Contract Logic

Contract logic is also the basis for approximation schemes. To stay with the example of JPY LIBOR contracts, interest rates are usually quoted as multiples of one basis point. However, in most circumstances one could safely introduce a coarsening approximation by postulating that rates are multiples of several basis points. One question that arises is what level of event space coarsening is acceptable in the sense of not altering the accuracy of the desired results appreciably.

Numeric Format

Financial data can always be reduced to a numerical format. An array of bits of a given finite length is called **numerical dataset**. Financial events can be casted in natural language but they exist and are stored as numerical datasets.

Propositional Logic

A **propositional logic** is given by a finite set $\mathcal{L}(\Omega)$ of functions $p(\omega)$ defined for $\omega \in \Omega$ and taking two values: **true** or **false**. A logic is assumed to be closed under three operations denoted with $\wedge$ (read **and**), $\vee$ (read **or**) and $\neg$ (read **not**). These logical operators are defined by the following table:

$p_1(\omega)$	$p_2(\omega)$	$(p_1 \wedge p_2)(\omega)$	$(p_1 \vee p_2)(\omega)$	$(\neg p_1)(\omega)$
true	**true**	**true**	**true**	**false**
true	**false**	**false**	**true**	**false**
false	**true**	**false**	**true**	**true**
false	**false**	**false**	**false**	**true**

Implication

Let $\mathcal{L}$ be a logic over the universe Ω. A proposition $p \in \mathcal{L}$ is said to **imply** a proposition $q \in \mathcal{L}$, written $p \longrightarrow q$ or $q \longleftarrow p$, if whenever $p(\omega) =$ **true** one also has that $q(\omega) =$ **true**. The propositions p and q are called **equivalent**, written $p \leftrightarrow q$, whenever they imply each other.

Elementary properties

If p_1, p_2 are two propositions in a logic $\mathcal{L}$, the following properties hold:

(i) **principle of the excluded middle:** $p_1 \vee \neg p_1 =$ **true** and $p_1 \wedge \neg p_1 =$ **false**;

(ii) **associativity:** $p_1 \vee (p_2 \vee p_3) = (p_1 \vee p_2) \vee p_3$ and $p_1 \wedge (p_2 \wedge p_3) = (p_1 \wedge p_2) \wedge p_3$;

(iii) **commutativity:** $p_1 \vee p_2 = p_2 \vee p_1$ and $p_1 \wedge p_2 = p_2 \wedge p_1$;

(iv) **absorption:** $p_1 \vee (p_1 \wedge p_2) = p_1$ $p_1 \wedge (p_1 \vee p_2) = p_1$;

(v) **distributivity:** $p_1 \vee (p_2 \wedge p_3) = (p_1 \vee p_2) \wedge (p_1 \vee p_3)$ and $p_1 \wedge (p_2 \vee p_3) = (p_1 \wedge p_2) \vee (p_1 \wedge p_3)$.

(v) **equivalence:** if $p_1 \leftrightarrow p_2$, then $p_1(\omega) = p_2(\omega)$ for all $\omega \in \Omega$.

An Example

An example of a verifiable proposition is

"The stock price S today is worth \$100".

There is of course nothing special about the figure of \$100. In general, one needs to assume that the set of prices any given asset can take a finite set of values. In fact currency is issued in integer multiples of cents. Furthermore, it is practically always possible to find an upper limit for stock prices that one can reasonably assume is never exceeded. If this limit is set at \$100,000, one is led to consider the following elementary verifiable propositions:

$$p_i := \text{"The stock price } S \text{ equals } S_i,$$

where $\{S_i = 0.01 \cdot i,\ i = 0, 1, ...10^7\}$ is the set of all possible stock price values.

Contract Logic

Given a family of financial contracts, one can speak of the logic $\mathcal{L}$ which is spanned by all propositions used to formulate the contracts. Interestingly, a logic determines a set which is typically much smaller than the universe Ω and contains only the information relevant to the specified contracts.

Atomic Propositions

Let $\mathcal{L}$ be a logic. A proposition $y \in \mathcal{L}$ is called atomic if it is not identically false, i.e. $y \neq$ **false**, and if there is no other proposition $p \in \mathcal{L}$ implied by y, i.e. such that $y \to p$. Let $\Lambda(\mathcal{L})$ be the set of all atomic propositions.

Atomic Propositions

Theorem. A generic proposition $p \in \mathcal{L}$ can be represented as follows:

$$p = y_1 \vee ... \vee y_n \tag{1}$$

where $\{y_1, ...y_n\}$ is the set of all atomic propositions in $\Lambda(\mathcal{L})$ which are implied by p.

Atomic Propositions

It suffices to prove that $p \leftrightarrow y_1 \vee ... \vee y_n$. It is obvious that $p \rightarrow y_1 \vee ... \vee y_n$. Do prove that $y_1 \vee ... \vee y_n \leftarrow p$, assume this is not the case and consider the proposition

$$y_0 = p \wedge \neg(y_1 \vee ... \vee y_n) \tag{2}$$

Then $y_0 \neq$ **false** and $p \rightarrow y_0$. Moreover, $y_0 \wedge y_i =$ **false** for all $i = 1, ...n$. Either y_0 is atomic itself or it will imply an atomic proposition $y_0' \neq$ **false**. But this contradicts the assumption that $y_1, ...y_n$ is the set of all atomic propositions implied by p.

Set Theoretical Representation

Let $\mathcal{L}$ be a logic over the space of events Ω and let $\Lambda(\mathcal{L})$ be the set of atomic propositions. If $p, q \in \mathcal{L}$, we introduce the following **set-theoretical terminology**:

- If y is an element of $A(p)$, one writes $y \in A(p)$. Otherwise $y \notin A(p)$;
- The empty set or null set is the set of no elements. It corresponds to the proposition identically **false** and is denoted by $\varnothing$;
- $\backsim A(p)$ is the set of all the elements in the universe X which are not in A;
- $C = A(p) \cup A(q)$ is the set such that $y \in C$ if and only if either $y \in A$ or $y \in A(q)$;
- $C = A(p) \cap A(q)$ is the set such that $y \in C$ if and only if $x \in A(p)$ and $x \in A(q)$;
- We say that $A(p) \subset A(q)$ if and only if $y \in A(q)$ whenever

Syllogisms

In the first logic treatise in history, the *Organon*, Aristotele studies syllogisms. A prototypical syllogism would be the following:

P_1 *Men are mortal*

P_2 *Socrates is a man*

C *If* P_1 *and* P_2 *are true, then Socrates is mortal.*

Syllogisms

P_1 is called the major premise, P_2 the minor premise and C is the conclusion. One should notice that the pattern of the argument is general as one may replace the terms in the proposition *men, mortal, Socrates* with other terms such as *cats, mammals, Sylvester*. In this case, if Sylvester is a cat then one can draw the conclusion that Sylvester is a mammal. Otherwise, if Sylvester is a parrot, the minor premise is invalid and the conclusion cannot be drawn.

Syllogisms

The two premises of the syllogism can be regarded as two propositions. To do so, one may define as event a triple of terms such as (Socrates, man, mortal) or (Sylvester, cat, mammal). The two premises may be either true or false depending on which terms are chosen to parameterize their content. The essence of the syllogism is the statement that if the terms are chosen so that the first two propositions are true, then for that choice of terms the third statement is also true.

Set Theoretical Representation

The set theoretical interpretation of Aristotelic logic is very natural. In the major premise, one considers the set of all men and declares that this set is contained in the set of all mortal beings. In the minor premise, one considers the set containing just Socrates as an individual and declares that this single element set is contained in the set of men. P2 states that the set made up of the individual Socrates alone is contained in the set of all men. This is an example of an atomic proposition. The conclusion is that since the set containing only Socrates is contained in the set of all men, and since the set of men is contained in the set of mortal beings, also Socrates is mortal.

More Syllogisms

If $p, q \in \mathcal{L}$, the following expressions are called **predicates**:

- $P_1 := p \to q$,
- $P_2 := p \wedge q = \textbf{false}$,
- $P_3 := p \wedge q \neq \textbf{false}$,
- $P_4 := \neg p \wedge q \neq \textbf{false}$.

A syllogism is an implication of the form $(P_i, P_j) \to P_k$. Such an implication is valid only for selected triples $i, j, k = 1, 2, 3, 4$.

More Syllogisms in Set Theoretical Language

In set-theoretical language, the above conditions are written as follows:

- $A(q) \subset A(p)$
- $A(p) \cap A(q) = \emptyset$
- $A(p) \cap A(q) \neq \emptyset$
- $\backsim A(p) \cap A(q) \neq \emptyset$

Temporal Modal Logic

Consider an elementary time interval $\delta t > 0$ and the family of time points $t_i = t_0 + i\delta t$ where $i = 0, 1, 2, ...$ and $t_0 \in \mathbb{R}$ is a fixed start time.

For each time t_i, consider also a propositional logic spanned by a set of atomic (mutually exclusive) propositions Λ_i. We assume that the size of the set Λ_i is finite. For notational convenience, I also assume that the size is constant and equal to d, i.e. $\Lambda_i = \{0, ...d - 1\}$.

Non Anticipatory Functionals

Let $\mathcal{P}(\Lambda)$ denote the set of all paths $\gamma = (y_i)_{i=0,1,\ldots}$ with $y_i \in \Lambda$. A function $\phi(\gamma, y, t_i)$ where $\gamma \in \mathcal{P}(\Lambda), y \in \Lambda, i = 0, 1, \ldots$ is called *non-anticipatory* if

$$\phi(\gamma, t_i) = \phi(\gamma', t_i) \tag{3}$$

whenever $\gamma_k = \gamma'_k$ for all $k \leq i$.

Path-spaces and Processes

A **path-space** $\mathcal{P}(\Lambda, k)$ is characterized by a sequence of *incidence matrices* given by non-anticipatory functions $k(\gamma, y; t_i)$ where $\gamma \in \mathcal{P}(\Lambda), y \in \Lambda, i = 0, 1, ...,$ taking only values 0 and 1.
A **real valued process adapted to the path-space** $\mathcal{P}'(\Lambda)$ is given by a real valued non-anticipatory function $A(\gamma, t_i)$.

Numeraire Assets

A numeraire is a positive valued process $g(\gamma, t_i) > 0$.
An example of a numeraire would be given by the price of a commodity with negligible carry costs such as, for instance, gold.
A second example of a numeraire is defined through a positive valued process $r(\gamma, t_i)$ interpreted as a **short rate process**. The corresponding numeraire is the **money market account** process given by

$$M(\gamma, t_i) = (1 + \delta t r(\gamma_0, t_0)) \quad \ldots \quad (1 + \delta t r(\gamma_{i-1}, t_{i-1})). \tag{4}$$

No-Arbitrage Condition

Let $\mathcal{K}(\Lambda)$ be a pathspace characterized by the incidence matrices $k(\gamma, y; t_i)$ and let g be a numeraire price process. Let $A_s(\gamma, t_i)$ be a family of adapted processes indexed by $s = 1, ...M$ with $M > 0$ on the time interval $t \in \{t_0, ...t_j = t_0 + j\delta t\}$ where $j > 0$. The family $A_s(\gamma, t_i)$ defines a g-**coherent family** if for any $t_i \in [t_0, t_j]$ and any set of coefficients $\zeta_s, s = 1, ...M$, the following property holds: if γ is a path for which $k(\gamma_i, \gamma_{i+1}; t_i) = 1$ and

$$\frac{1}{g_s(\gamma, t_{i+1})} \sum_s \zeta_s A_s(\gamma, t_{i+1}) - \frac{1}{g_s(\gamma, t_i)} \sum_s \zeta_s A_s(\gamma, t_i) > 0 \quad (5)$$

then there is a second path γ' such that $k(\gamma'_i, \gamma'_{i+1}; t_i) = 1$ and

$$\frac{1}{g_s(\gamma, t_{i+1})} \sum_s \zeta_s A_s(\gamma, t_{i+1}) - \frac{1}{g_s(\gamma, t_i)} \sum_s \zeta_s A_s(\gamma, t_i) < 0. \quad (6)$$

No-Arbitrage Condition

- The financial interpretation of the definition of coherent family of processes is that they represent *asset price processes*.
- Coherence describes a situation when one holds a portfolio at time t_i and compares the value this portfolio has at time t_{i+1} vis-a-vis the value it would have if it was entirely converted into the numeraire asset at time t_i. The condition states that whenever holding the portfolio achieves a gain with respect to holding just the numeraire in a possible scenario at time t_{i+1}, then there needs to be another scenario whereby one would instead register a loss.
- In other words, **financial profit is possible only if one accepts also the possibility of a loss**. This is a fundamental principle of finance with ancient roots.