

# DISCRETE CREDIT BARRIER MODELS

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ABSTRACT. The model introduced in this article is designed to provide a consistent representation for both the real-world and pricing measures for the credit process. We find that good agreement with historical and market data can be achieved across all credit ratings simultaneously. The model is characterized by an underlying stochastic process that takes on values on a discrete lattice and represents credit quality. Rating transitions are associated to barrier crossings and default events are associated with an absorbing state. The stochastic process has state dependent volatility and jumps which are estimated by using empirical migration and default rates. A risk-neutralizing drift is estimated to consistently match the average spread curves corresponding to all the various ratings.

## 1. INTRODUCTION

The credit model developed in this paper is specified with respect to aggregate data. That is, the credit quality of individual obligors is modelled within the wider context of all firms that issue debt. There are various advantages to working in the aggregate approach: the model is cohesive and intuitive, insight into the individual processes can be gained when examined in the wider context, and a larger class of derivative contracts can be priced.

In the statistical measure the model is calibrated to fit historical average rating transition and default probabilities. The underlying stochastic process in the model represents credit quality and rating transitions are associated to barrier crossings. While the volatility structure is shared for all of the credit quality processes, the difference for each process is in the starting point. In this way the aggregate flow of credit quality is modelled. In order to achieve a consistent representation for both the statistical measure and the pricing measure, a risk-neutralizing drift is introduced to the statistical measure to match market average spread rates on a rating basis rather than an individual firm basis.

The model which we present in this paper and which we call the credit barrier model is considered to be a hybrid of a structural credit model and a reduced form credit model, the two traditional classes of credit risk models. Structural models were initiated with the work of Black and Scholes (1973) and Merton (1974). Many variants have been proposed, examples of which are: Kim, Ramaswamy and Sundaresan (1993), Nielsen, Saa-Requejo and Santa-Clara (1993) and Longstaff and Schwartz (1995). This class of models attempts to directly model the capital structure of a firm, with the default of debt as an endogenous event. The firm value is modelled as a continuous diffusion and default occurs if the firm value falls below a pre-specified barrier that represents the firm's liabilities. Calibration of these models pose a problem as the firm value and liability structure is information that is often not disclosed. In addition, since the underlying stochastic process is a diffusion, default probabilities in the short term will be very low unless the firm value starts near the barrier. This leads to zero short maturity spreads, a feature that is in disagreement with market observed spreads.

Artzner and Delbaen (1995), Jarrow and Turnbull (1995) and Duffie and Singleton (1999) are early examples of reduced form models. In this class of models, default is modelled as an exogenous process. The intensity rate of a Poisson process is adjusted in order that default

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probabilities occur in accordance to market spread rates. While market spread rates can be reproduced, default is an unanticipated event, contrary to situations where there is a gradual degradation in credit quality.

Hybrids of structural and reduced form are sometimes referred to as incomplete information models. These are hybrids in the sense that default can occur as a jump to default, similar to the case of reduced form models, or they can occur as a process crosses a barrier in a continuous manner.

Credit models that use the credit rating classes of firms such as Moody's and Standard and Poor's can be seen as incomplete information models. In Jarrow, Lando and Turnbull (1997) the empirical transition matrix is taken as the Markov transition matrix. Thus, firms can either step down in rating towards default or just skip several ratings straight to default. Gordy and Heitfield (2001) proposed a model in which the credit rating was a continuous process, but found that transitions could not be modelled properly since the processes they used lacked the necessary kurtosis. Douady and Jeanblanc (2002) have proposed a theoretical model based on the same idea, but using a jump diffusion to model the credit quality. The present model gives a diffusion process that is subordinated on a discontinuous, random time process, and we find that the model can be calibrated with good agreement to market data.

A new class of credit barrier models was introduced by the authors in Albanese, Campolieti, Chen and Zavidonov (2003) and Albanese and Chen (2004a) in an attempt to reconcile the real-world and the risk-neutral measure. In the real-world measure, the driving process is identified with the observable credit rating so that empirical credit migrations and defaults can be estimated. From this, a time-dependent default barrier is used to transform to the risk-neutral measure in order that spread curves can be estimated. In this way, the estimation framework is extended to include a more comprehensive set of statistical data such as historical migration rates, default frequencies over several time horizons and aggregate spread curves across all ratings. Within the extended framework, one obtains metrics for relative liquidity spreads across credit ratings. One also obtains a new methodology to extrapolate implied migration rates.

In the continuous framework of Albanese et al. (2003), the probability transition densities for the diffusion process in the statistical measure could be readily computed using available numerical libraries for special functions. In addition, the default probabilities could be rapidly computed even under the stochastic time change, as this involved just a single integral. However, the transition probabilities between credit rating classes involved an integration over the space variable and the time variable when applying the stochastic time change. The situation in the pricing measure was even worse, as the transition probability density could only be found by numerically solving a partial differential equation and then integrating that density against the gamma density to apply the stochastic time change.

Thus, in order to have a more robust model Albanese and Chen (2004b) utilised a discretization of the space variable in the statistical measure that allows the calculation of transition probabilities as finite sums of orthogonal polynomials of a discrete variable. The functional form of the transition probabilities is such that applying a stochastic time change does not increase the computational load, as was the case in the continuous framework. In addition, the computational load in calculating transition probabilities is independent of time difference between the initial and final state. The methods involved in the discretisation of the state space follow in the spirit of early works by Ledermann and Reuter (1954) and Karlin and McGregor (1957), who studied the spectral theory of birth-death processes. More recently, Albanese and Kuznetsov (2003) have applied these lattice models to mathematical finance.

While Albanese and Chen (2004b) presented a possibility for the change from the real-world to the risk-neutral measure, a different method was found and will be presented here. This new method is more robust and gives the model flexibility in correlating different processes in the risk-neutral measure.

We proceed as follows: section 2 contains the formulation of the discrete process in the real-world measure and presents the method of risk-neutralization. Section 3 gives a numerical

example for the discrete process to compare with the example for the continuous process given in Albanese et al. (2003). Section 4 concludes.

## 2. MODEL DESCRIPTION

The full model includes both a real-world process and a risk-neutral process. The latter is obtained from the former by the inclusion of a drift. Both processes take place on a finite lattice, with different adjacent groupings of lattice points corresponding to different credit ratings. At one end of the lattice exists an absorbing state with absorption corresponding to default. With this formulation it is possible to calculate transition probabilities between ratings and default probabilities, given an initial node on the lattice.

The real-world process is calibrated to match historically observed default probabilities and rating transition probabilities. A drift is added to the real-world process to obtain the risk-neutral process. Default probabilities of the risk-neutral process lead to credit spreads and the drift is calibrated in order that these spreads match market spread rates over all ratings and maturities.

In this section we fully specify the real-world process and also the risk-neutralizing drift.

**2.1. Real-world process.** The construction of the real-world process begins with an underlying process which we call the Hahn process by virtue of the fact that it can be described in terms of Hahn polynomials. The use of Hahn polynomials makes the calculation of the kernels numerically efficient. The Hahn process does not have all of the desired properties that we wish the real-world process to have, so a measure change and transformation are performed to obtain the required properties without sacrificing computability. In addition to this, jumps are added in the form of a stochastic time change in order to better match historical data.

**2.1.1. Underlying process.** The Hahn process takes values on the discrete lattice  $\Lambda = \{0, 1, 2, \dots, N\}$ , for some positive integer  $N$ . In what follows, we fix the value of  $N$  throughout the description of the model. However, the value of  $N$  can be adjusted in the calibration of the model in order to balance refinement and numerical efficiency.

The Markov generator  $\mathcal{L}$  of the Hahn process can be defined by the way it acts on functions  $f$  with domain  $\Lambda$ :

$$(1) \quad \mathcal{L}f(x) = \begin{cases} N(\alpha + 1)(f(1) - f(0)) & \text{if } x = 0; \\ D(x)\Delta f(x) + [B(x) - D(x)]\nabla_+ f(x) & \text{if } 0 < x < N; \\ N(\beta + 1)(f(N - 1) - f(N)) & \text{if } x = N, \end{cases}$$

where  $\Delta f(x) = f(x + 1) + f(x - 1) - 2f(x)$  and  $\nabla_+ f(x) = f(x + 1) - f(x)$  and:

$$\begin{aligned} B(x) &= (N - x)(x + \alpha + 1) \\ D(x) &= x(N + \beta + 1 - x) \quad x \in \Lambda. \end{aligned}$$

From (1) we see that  $\mathcal{L}$  can be represented by a tri-diagonal matrix  $L$ :

$$(2) \quad L = \begin{pmatrix} b_0 & a_0 & 0 & \cdots & \cdots & 0 \\ c_1 & b_1 & a_1 & 0 & \cdots & 0 \\ 0 & c_2 & b_2 & a_2 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & c_{N-1} & b_{N-1} & a_{N-1} \\ 0 & 0 & \cdots & 0 & c_N & b_N \end{pmatrix}$$

where

$$\begin{aligned} a_x &= B(x), \quad x = 0, \dots, N - 1 \\ b_x &= -B(x) - D(x), \quad x = 0, \dots, N \\ c_x &= D(x), \quad x = 1, \dots, N. \end{aligned}$$

Written in matrix form, the condition for a process to be one in which probability is conserved is for all of the rows to sum to zero. While this clearly holds on the interior points, it is also true for the zero-th and  $N^{\text{th}}$  row since  $D(0) = 0$  and  $B(N) = 0$  respectively. However, for a stochastic process that would be appropriate to model credit risk, we require absorption at zero to emulate default. In addition, we require the real-world process to be driftless on the interior of the lattice. Both of these properties can be acquired by the introduction of a carefully chosen measure change and transformation.

*2.1.2. Constructing a driftless process.* The credit quality process is obtained from the Hahn process by use of a measure change and a transformation from the lattice  $\Lambda$  to a lattice  $\tilde{\Lambda}$ . The measure change and transformation are specified in such a way that the credit quality process can be described by a Markov generator  $\tilde{\mathcal{L}}$  that can be easily obtained from  $\mathcal{L}$ .

The process described by  $\tilde{\mathcal{L}}$  will be appropriate as a credit quality process since there will be absorption at the lower boundary to emulate default, and the process will be driftless to reflect the desire of the credit rating agencies to have a system of rating debt consistently in time and independent of the economic cycle. A process that moves lower toward the absorbing boundary represents a degradation in credit quality, while a process that moves higher away from the absorbing boundary represents an enhancement in credit quality.

Here, we specify the measure change and transformation.

Given a real number  $\rho$ , we seek two linearly independent functions  $f_1$  and  $f_2$  on the lattice  $\Lambda$  that satisfy:

$$(3) \quad \mathcal{L}f_i(x) = \rho f_i(x) \quad x = 1, \dots, N-1; \quad i = 1, 2.$$

If we choose two sets of terminal points  $\{f_1(N-1), f_1(N)\}$  and  $\{f_2(N-1), f_2(N)\}$  such that:

$$\frac{f_1(N)}{f_2(N)} \neq \frac{f_1(N-1)}{f_2(N-1)}$$

then we can use the recurrence relation:

$$D(x)f_i(x-1) - [B(x) + D(x)]f_i(x) + B(x)f_i(x+1) = \rho f_i(x)$$

obtained from (1) to iterate  $f_1$  and  $f_2$  backwards to obtain the required linearly independent functions on all of  $\Lambda$ . Note that in order to use this recurrence relation arbitrary precision calculations are required. Assume that two coefficients  $c_1$  and  $c_2$  can be chosen such that the function

$$g = c_1 f_1 + c_2 f_2$$

is strictly positive on  $\Lambda$ . In addition, we choose two coefficients  $c_3$  and  $c_4$  to define the transformation  $y = Y(x)$  as:

$$Y = \frac{c_3 f_1 + c_4 f_2}{g} = \frac{c_3 f_1(0) + c_4 f_2(0)}{g(0)}.$$

From this, we see why  $f_1$  and  $f_2$  are required to be linearly independent: if they were not then the transformation  $Y$  would be singular. Finally, we define the Markov generator  $\tilde{\mathcal{L}}$  for the credit quality process on the lattice  $\tilde{\Lambda} = \{Y(0), \dots, Y(N)\}$  by:

$$\tilde{\mathcal{L}} = g^{-1}(X)(\mathcal{L} - \rho)g(X)$$

where  $X$  is the inverse of  $Y$ . We can represent  $\tilde{\mathcal{L}}$  by the matrix  $\tilde{L}$  with  $(x_0, x)$  element:

$$\tilde{L}_{x_0, x} = g(x_0)^{-1}[L_{x_0, x} - \rho \delta_{x_0, x}]g(x),$$

where  $\delta$  is the  $N \times N$  identity matrix. It is easy to verify from this expression that for a process  $y_t$  with  $\tilde{\mathcal{L}}$  as its Markov generator,  $y_t$  is driftless on the interior of the lattice. In general, the last row of this matrix will not sum to zero and there will either be probability leakage or a probability source at  $N$ . However, we can choose, say,  $c_2$  in terms of  $c_1, f_1(N-1), f_1(N), f_2(N-1)$  and  $f_2(N)$  such that

$$g(N)^{-1}L_{N, N-1}g(N-1) + L_{N, N} - \rho = 0.$$

If this is satisfied, then probability will be conserved at the upper boundary and there will be probability absorption at the lower boundary depending on the value of  $\rho$ . Due to the difference equation, the range of  $\rho$  for which there is absorption at zero can be determined by finding the negative regions of a degree  $N$  polynomial with coefficients dependent on the parameters  $\alpha$  and  $\beta$ . However, for our purposes it is far more efficient to merely try different values of  $\rho$  and disregard any that do not lead to absorption at zero.

2.1.3. *Representation of the kernel.* Here, we describe the computation of the kernel which is seen to be numerically efficient.

The Hahn polynomials  $\{H_n\}_{n=0}^N$  are a set of orthogonal polynomials of a discrete variable, as described in the appendix. By comparing (1) to equation (A-3) we see that the Hahn polynomials are eigenvectors of  $\mathcal{L}$  with eigenvalues given by (A-4):

$$\mathcal{L}H_n(x) = \lambda_n H_n(x).$$

This holds true for all  $x$ , even on the boundary, since  $D(0) = 0$  and  $B(N) = 0$ . Thus, the set of vectors given by:

$$\varphi_n(x) = d_n H_n(x)$$

where the  $d_n$  are defined in (A-7) are eigenvectors of  $\mathcal{L}$  with eigenvalues  $\lambda_n$ , are orthonormal on  $\Lambda$  with respect to the weight function  $w$ , given in (A-6).

We see that the eigenvectors of  $\tilde{\mathcal{L}}$  are just given by  $\psi_n = g^{-1}(X)\varphi_n(X)$  with eigenvalue  $\lambda_n - \rho$ . Indeed,

$$\begin{aligned} \tilde{\mathcal{L}}g^{-1}(X)\varphi_n(X) &= g^{-1}(X)(\mathcal{L} - \rho)\varphi_n(X) \\ &= (\lambda_n - \rho)g^{-1}(X)\varphi_n(X). \end{aligned}$$

Furthermore, these eigenvectors are orthonormal on the lattice  $\tilde{\Lambda}$  with respect to the weight  $\tilde{w} = w(X)g^2(X)$ :

$$\sum_{y \in \tilde{\Lambda}} \psi_m(y)\psi_n(y)\tilde{w}(y) = \delta_{m,n}.$$

Recall that given the Markov generator  $\tilde{\mathcal{L}}$  of the credit quality process, the probability kernel  $U_t(y_0, y)$  will satisfy the backward Kolmogorov equation:

$$(4) \quad \frac{\partial U_t(y_0, y)}{\partial t} = \mathcal{L}U_t(y_0, y)$$

with initial condition:

$$(5) \quad U_0(y_0, y) = \delta_{y_0, y},$$

where  $y_0, y \in \tilde{\Lambda}$ . The solution to (4) and (5) is:

$$(6) \quad U_t(y_0, y) = e^{t\tilde{\mathcal{L}}}\delta_{y_0, y}.$$

Now,  $\delta_{y_0, y}$  can be written as:

$$\delta_{y_0, y} = \tilde{w}(y) \sum_n \psi_n(y_0)\psi_n(y)$$

so that (6) becomes:

$$(7) \quad U_t(y_0, y) = \tilde{w}(y) \sum_n e^{(\lambda_n - \rho)t} \psi_n(y_0)\psi_n(y).$$

We note that this expression for the transition probabilities differs fundamentally with those found previously in, for example Ledermann and Reuter (1954), Karlin and McGregor (1957), Karlin and McGregor (1961) and van Doorn (2003). In those, the transition probabilities are expressed as sums over the arguments of orthogonal polynomials, rather than sums over the indices of the polynomials as we have.

2.1.4. *Credit quality process.* The Markov generator is defined such that probability is preserved at the upper boundary  $y_N = 1$ , while at the lower boundary  $y_0 = 0$  there is an ignored, absorbing state  $y_{-1}$ . Absorption into the ignored state is considered as the occurrence of default. The interpretation of this stochastic process into a credit quality process is as follows. For a credit rating system with  $M$  different ratings, we sub-divide the nodes  $\{y_0, \dots, y_N\}$  into  $M$  groups of adjacent nodes:  $I_i = \{y_{a_{i-1}}, \dots, y_{a_i-1}\}$  for  $i = 1, \dots, M$ , where  $0 = a_0 < a_1 < \dots < a_M = N+1$ . The groups of nodes  $I_i$  correspond to the various credit classes so that if a process is in  $I_i$  at time  $t$ , then it is said to have a credit rating of  $i$ . For each  $i$ , choose a node  $b_i \in I_i$  to serve as an initial node. Then, the conditional transition probability  $p_{ij}(t)$  that an obligor with given initial rating  $i$  at time 0 will have a rating  $j$  at a later time  $t > 0$  is just:

$$(8) \quad p_{ij}(t) = \sum_{k=a_{j-1}}^{a_j-1} U_t(b_i, y_k).$$

The probability that starting from the initial rating  $i$  and reaching a state of default by time  $t$  is given by the following difference:

$$(9) \quad p_i^D(t) = 1 - \sum_{k=0}^N U_t(b_i, y_k).$$

As with the continuous diffusion model in Albanese et al. (2003), we were unable to fit such a lattice model to the historical migration rates. The problem we encountered was that although transition probabilities  $p_{ij}(t)$  for nearest rating migrations can be well reproduced, it is not possible to simultaneously fit migrations involving a rating change of two or three levels and to accurately reproduce default probabilities across credit ratings.

2.1.5. *Adding jumps.* The difficulties above can be overcome by allowing for jumps. In the lattice process described thus far, only nearest neighbour transitions are allowed. A jump in this case refers to a transition to a node that is not adjacent to the starting node. To include jumps, we use stochastic time changes similarly to what is done in the Variance-Gamma (VG) model by Madan, Carr and Chang (1998). There, jumps are added to continuous geometric Brownian motion (GBM) by evaluating GBM at a random time given by a gamma process. Since gamma processes increase in a discontinuous manner, subordinating on a gamma process leads to jumps in the original process.

In our case, a process  $V_t$  with jumps is obtained by evaluating  $y_t$  at a random time given by a gamma process  $\gamma(t, 1, \nu)$ . That is,

$$(10) \quad V_t = y_{\gamma(t, 1, \nu)},$$

where  $\nu$  is called the variance rate and has the dimension of time, and  $y_t$  has (7) as its probability kernel.

The process  $\gamma(t, 1, \nu)$  can be interpreted as a mapping from calendar time  $t$  to a financial time  $\tau$ . Financial time can be defined, for example as the total number of transactions up to a certain calendar time, or the total volume of transactions. Jumps in financial time then reflect an occurrence of a transaction.

A gamma time change can be applied to obtain a time-changed probability kernel by integrating the probability kernel without jumps against the probability density function of the gamma distribution:

$$(11) \quad \tilde{U}_t(y, y') = \int_0^\infty U_s(y, y') \tilde{\Gamma}(s, t) ds,$$

with

$$(12) \quad \tilde{\Gamma}(s, t) = \frac{s^{t/\nu-1} e^{-s/\nu}}{\Gamma(t/\nu) \nu^{t/\nu}}$$

the gamma distribution and  $\Gamma(x)$  the Gamma function. But it is seen from equation (7) that this is just a Laplace transform, so that the time changed probability kernel can be expressed

as:

$$(13) \quad \bar{U}_t(y, y') = \tilde{w}(y') \sum_{n=0}^N e^{-\phi(\rho - \lambda_n)t} \psi_n(y) \psi_n(y')$$

where

$$(14) \quad \phi(\lambda) = \frac{\mu^2}{\nu} \log \left( 1 + \lambda \frac{\nu}{\mu} \right)$$

is the Laplace exponent of the gamma process.

Under the stochastic time change, the transition probabilities are:

$$(15) \quad \tilde{p}_{ij}(t) = \sum_{k=a_{j-1}}^{a_j-1} \tilde{U}_t(b_i, y_k),$$

and the default probabilities are:

$$(16) \quad \tilde{p}_i^D(t) = 1 - \sum_{k=0}^N \tilde{U}_t(b_i, y_k).$$

**2.2. Risk-neutralizing Drift.** In a departure from the continuous model given in Albanese et al. (2003), the risk-neutralizing drift is applied after the stochastic time-change. In this framework, we first discretise the time variable into steps  $\Delta t > 0$  so that the process is considered only at times  $t_i = i\Delta t$  for  $i = 0, 1, \dots$ . Thus, we need only consider the one-step probability kernel  $\tilde{U}_{\Delta t}(y, y')$  in the real-world measure.

For positive functions of three variables  $g(y, y'; t_i)$ , define

$$(17) \quad \bar{g}(y; t_i) = \sum_{y'} \tilde{U}_{\Delta t}(y, y') g(y, y'; t_i).$$

Here, the summation includes the state  $y_{-1}$  with  $\tilde{U}_{\Delta t}(y, y_{-1}) = 1 - \sum_{k=0}^N \tilde{U}_{\Delta t}(y, y_k)$ . The transformed kernel for transitions between  $t_i$  and  $t_{i+1}$  is then given by

$$(18) \quad \tilde{U}_{t_i, t_{i+1}}(y, y') = \frac{1}{\bar{g}(y, t)} \tilde{U}_{\Delta t}(y, y') g(y, y'; t_i).$$

Probability kernels for more than one time-step can be computed by composing these one-step kernels.

The function  $g(y, y', t_i)$  is chosen in order that spread curves are matched by the transformed default probabilities, given by:

$$(19) \quad \bar{p}^D(t_i) = 1 - \sum_{k=0}^N \bar{U}_{0, t_i}(y, y_k)$$

### 3. MODEL ESTIMATION

In this section a numerical example is given in which the real-world process is calibrated to historical default and transition probabilities, and the drift is calibrated so that the risk-neutral process matches average market spread rates. In addition, it is possible to calculate risk-neutral transition probabilities.

While this model is highly parameterized, the large number of parameters are indeed necessary to fit the complex processes of credit migrations for all rating classes simultaneously. An algorithm is described below that eases the burden of estimating all of the parameters.

**3.1. Real-world process.** For the historical data, we used the full one year transition probability matrix taken from Carty (1997). The matrix as provided contains an additional category labelled “Withdrawn Rating”. This category includes all issues that were rated at the beginning of the given time period but not rated at the end of the observation period, whether it was because the debt was retired, due to a lack of information available to Moody’s or any number of other reasons. If the debt was retired, then a withdrawn rating has no implication on the credit risk. On the other hand, if the rating was withdrawn due to a lack of information, this could be the result of a degradation in credit quality. Since the withdrawn rating gives no clear indication about the credit risk of the issuer, we adjusted the transition matrix to be conditional upon the issuer not having their rating withdrawn within the observation period.

For historical default probabilities, we used one, three and five year time periods. The additional time horizons were used for the default probabilities due to the relative importance of the default probabilities in credit risk models. The three and five year transition matrices were not used because the data was too noisy for the low probability events since the averaging was taken over a shorter time period (over 1983-1994 and 1983-1992 respectively for the three and five year transition matrices, as opposed to 1983-1996 for the one year transition matrix). As with the transition probabilities, the data was taken from Carty (1997) and is withdrawn rating adjusted.

Note that while the calibrations were performed for all of the 17 ratings, we generally show just a representative sampling of 7 of the ratings.

Under the real-world measure, the model was calibrated to minimize the the difference between the components  $\tilde{p}_{ij}(t)$  and  $\tilde{p}_i^D(t)$  of the model and the corresponding components from the historical data. For a lattice of size  $N$ , in order to fully specify the model it is necessary to set the free parameters  $\alpha, \beta, \rho$ , the variance rate of jumps  $\nu$ , a time-scaling factor  $\tau$ , the groupings of nodes  $I_i$  and the initial nodes  $b_i$ . In fact, the  $b_i$  were not restricted to the set of nodes  $\{y_k\}_{k=0}^N$  but were free to be between nodes, with probabilities then calculated as weighted sums of probabilities from adjacent nodes.

While optimising over such a large number of parameters seems a daunting numerical task, significant simplifications can be made. Given a set of parameters  $\alpha, \beta, \rho, \nu$  and  $\tau$ , default probabilities can be calculated by use of equation (16). So, for the lower credit ratings where the probability of default is significant, an approximation for the initial credit levels  $b_i$  can be made by matching the calculated model default probabilities with the historical default probabilities.

Suppose that the lowest  $i_0$  credit levels can be estimated in this way. With an approximation of  $b_1$  made for this parameter set, it is possible to make an approximation on  $a_1$ , since we only need  $a_1$  and  $b_1$  in order to determine  $\tilde{p}_{1,1}(1)$ . By stepping up a rating at each iteration, for  $i \leq i_0$ , we need only  $a_{i-1}, a_i$  and  $b_i$  to determine  $\tilde{p}_{i,i}(1)$ . The diagonal elements (corresponding to the probability that the credit ratings do not change) are used because they are always the largest elements of the transition matrix.

To this point, we have approximations on  $b_1, \dots, b_{i_0}$  and  $a_1, \dots, a_{i_0}$ . For  $b_{i_0+1}$ , we can calculate  $\tilde{p}_{i_0+1,j}(1)$  for  $j \leq i_0$ . Thus, an approximation of  $b_{i_0+1}$  can be made by matching these model transition probabilities with the historical probabilities. In turn,  $a_{i_0+1}$  can be approximated as before, by matching the diagonal element  $\tilde{p}_{i_0+1,i_0+1}(1)$ . Thus, we can iterate this process until all of the credit levels and barriers are estimated for this set of parameters.

Thus, given a set of parameters  $\alpha, \beta, \rho, \nu$  and  $\tau$  we can obtain a first approximation on the credit levels and barriers relatively easily. After finding a reasonable set of parameters, the approximations on the credit levels and barriers can be further refined to better fit the historical data.

We estimated our model under the real-world measure using historical credit migration rates over a one year time horizon and default rates over a one, three and five year time horizon. The quality of fit is depicted in Figures 1 and 2, using the parameter values  $N = 1000, \alpha = 1.2, \beta = -0.5, \rho = 0.016, \nu = 0.5$  and  $\tau = 0.0425$ .

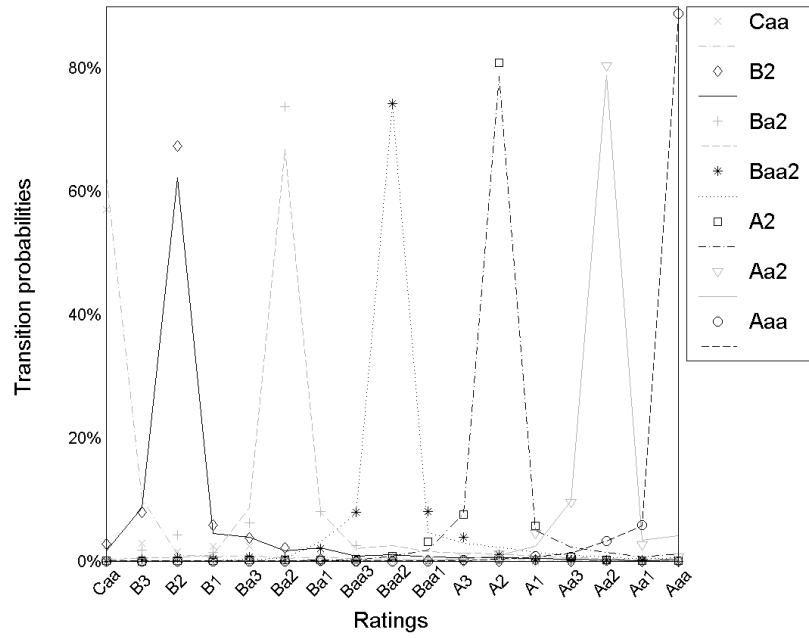


FIGURE 1. Comparison of model (lines) and historical (dots) one year transition probabilities. Historical transition probabilities are taken from Carty (1997) and are “Withdrawn Rating” adjusted.

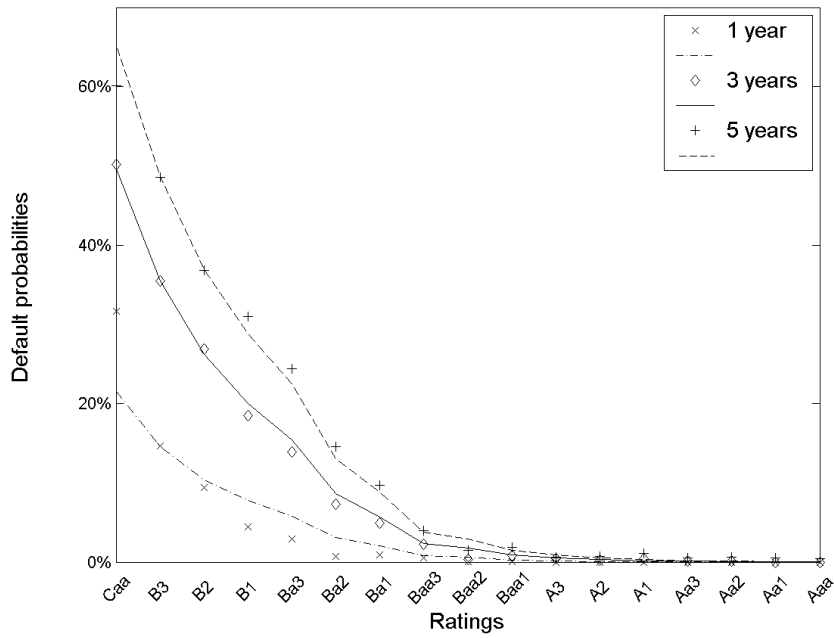


FIGURE 2. Comparison of model (lines) and historical (dots) default probabilities. Historical default probabilities are taken from Carty (1997) and are “Withdrawn Rating” adjusted.

We notice that the fit is quite good across all ratings and time horizons. A measure of the volatility taken as:

$$(20) \quad \sigma^2(y_i) = \lim_{\Delta t \rightarrow 0} \left[ \frac{P(y_{t+\Delta t} = y_{i+1} | y_t = y_i)(y_{i+1} - y_i)^2}{\Delta t} + \frac{P(y_{t+\Delta t} = y_{i-1} | y_t = y_i)(y_{i-1} - y_i)^2}{\Delta t} \right]$$

is shown in Figure 3. Notice that the volatility function depends on the credit rating and is higher for lower quality ratings. Note also the groupings of nodes for the different credit ratings.

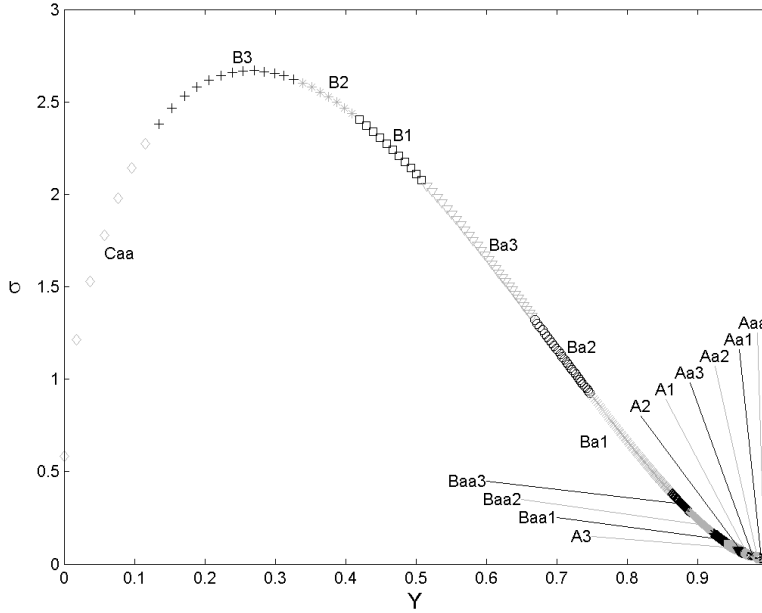


FIGURE 3. Measure of volatility given in Equation (20).

The variance rate we estimate out of credit migration data is  $\nu = 0.5$  years. This is a large value as compared to variance rates implied from statistical equity returns as found in Madan et al. (1998), indicating that large jump amplitudes are required to justify the observed transition probabilities. Since the jump amplitudes are affected by the volatility, we conclude that the intensity of jumps is not homogeneously distributed between credit ratings. Specifically, larger jumps are associated with the lower ratings.

**3.2. Risk-neutral process.** With the real-world model determined, we can add the risk-neutral drift through specification of the function  $g(y, y'; t)$ . The calibration of the  $g$  is formulated in terms of forward liquidity spreads. The forward liquidity spread  $c(T, T + \tau, i)$  for the period  $[T, T + \tau]$  and the average spread curve of rating  $i$ , computed with simple compounding over the period  $\tau$ , is defined so that

$$(21) \quad f^{\text{mkt}}(T, T + \tau, i) = f^{\text{mdl}}(T, T + \tau, i) + c(T, T + \tau, i)$$

where  $f^{\text{mkt}}(T, T + \tau, i)$  and  $f^{\text{mdl}}(T, T + \tau, i)$  are the market and model forward rates, respectively. Thus,  $g$  is calibrated in such a way that a weighted sum of the squares of  $c(T, T + \tau, i)$  over all ratings and over all relevant maturities is minimized. In addition, we apply the constraint that the forward liquidity spread is strictly positive for investment grade ratings.

Original Rating	Recovery Rate(%)
Aaa	68.34
Aa2	59.59
A2	60.63
Baa2	49.24
Ba2	39.05
B2	37.54
Caa	38.02

TABLE 1. Recovery rate for each credit rating. From Altman and Kishore (1998).

Rather than individual issue spread rates, the market forward rates that are used are derived from aggregate spread data averaged within each rating class.

The *credit spread* is the the portion of the total yield spread due to credit risk. It is shown in Huang and Huang (2002) that the credit spread is a small part of the yield spread in higher ratings and is a larger fraction in lower ratings. With the chosen  $g$ , this qualitative behaviour is reflected in the spread rates computed by the model, as shown in Figure 4. Note that we are taking the simple case of zero-coupon bonds and assuming a constant interest rate, so that cumulative probabilities of default and recovery rates imply a term structure for spread rates through the pricing formula:

$$(22) \quad e^{-s(t)t} = 1 - P(t)(1 - R)$$

where  $s(t)$  is the yield spread for maturity  $t$ ,  $P(t)$  is the probability of defaulting before  $t$  and  $R$  is the recovery rate. For the recovery rates, we use historical recovery rates that are rating dependent. These are displayed in Table 1.

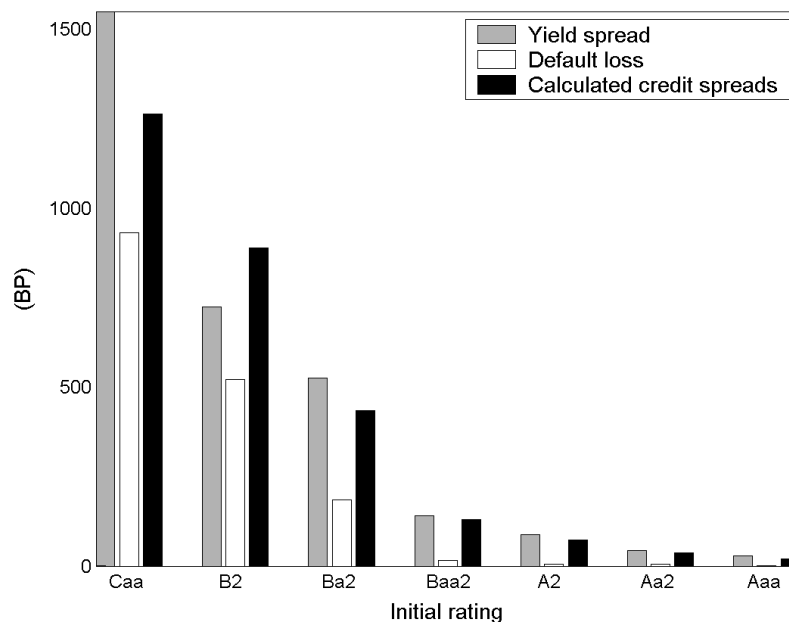


FIGURE 4. Yield spreads, default loss rates and calculated credit spreads for 5-Year bonds. Yield spreads are Bridge Evaluator Corporate Spreads for Industrials taken from <http://www.bondsonline.com> on Feb. 10, 2003.

Elton, Gruber, Agrawal and Mann (2001) find that the mid-point of effective state tax rates is 4.875%. Assuming zero-coupon bonds, this gives a tax-adjusted spread rate of:

$$(23) \quad s(t) - 4.875\%[s(t) + r(t)]$$

where  $r(t)$  is the treasury yield of maturity  $t$ . Figures 5 and 6 compare the tax-adjusted spread curves with the model derived spread curves assuming constant implied recovery rates. One can notice that the qualitative behaviour of the term structure of credit spreads is correctly reproduced by our model, with lower ratings corresponding to downward sloping curves and higher ratings corresponding to upward sloping and flatter profiles. The difference between the market and model credit spreads can be explained by the forward liquidity spreads, which are charted in Figure 7.

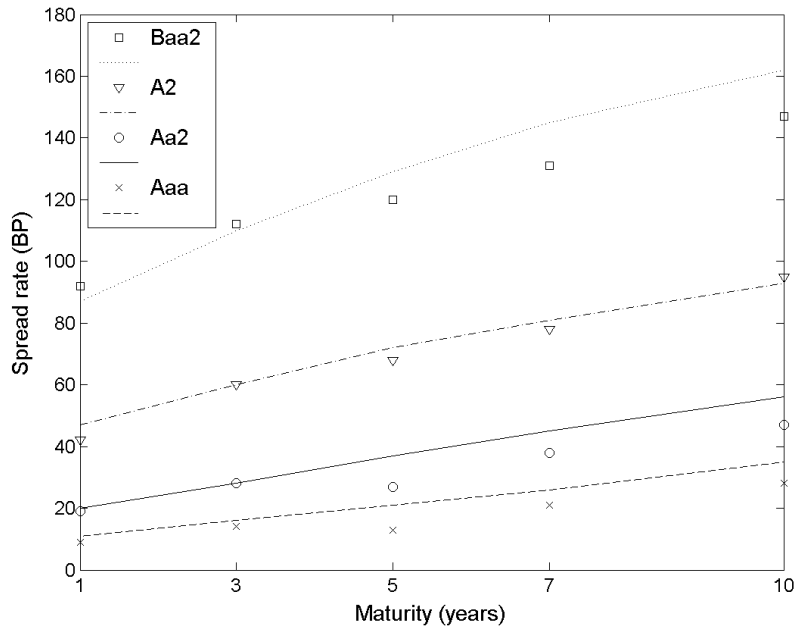


FIGURE 5. Comparison of the term structures for theoretical and tax-adjusted credit spreads for investment grade rated bonds.

**3.3. Risk-neutral transition probabilities.** With the risk neutralizing drift specified, we can calculate the risk-neutral transition probabilities by the formula:

$$(24) \quad \bar{p}_{ij}(t) = \sum_{k=a_j}^{a_{j+1}-1} \bar{U}_{0,t}(b_i, y_k).$$

We plot the ratio of risk-neutral to real-world transition probabilities in Figures 8 and 9 for one and five years, respectively.

Since risk-neutral default probabilities are greater than real-world default probabilities, we would expect risk-neutral downgrade probabilities to be greater than real-world downgrade probabilities, and risk-neutral upgrade probabilities to be less than real-world upgrade probabilities. It was shown in Albanese and Chen (2004a) that this behaviour is not found in credit rating models proposed by Jarrow et al. (1997) and Kijima and Komoribayashi (1998) but is displayed by the credit barrier model in the continuous framework. Figures 8 and 9 show that this behaviour is also found in the credit barrier model in the discrete framework.

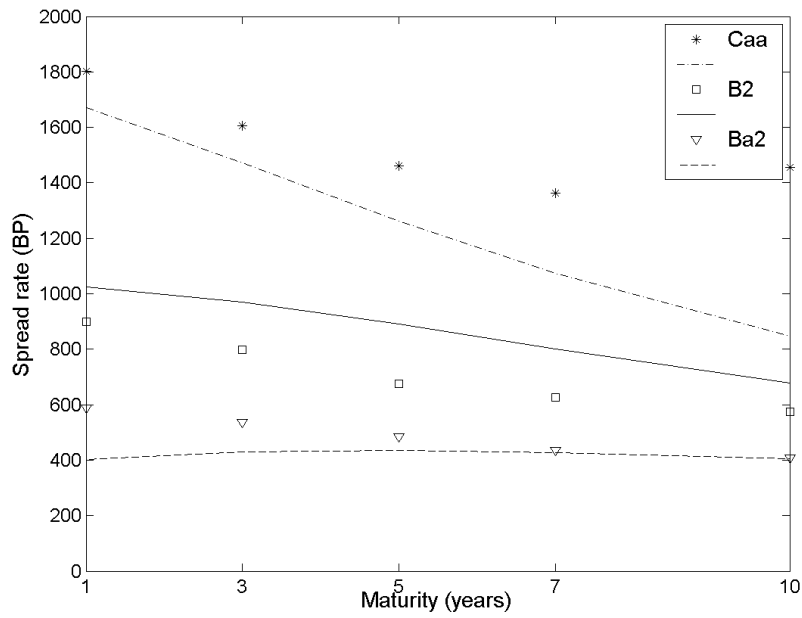


FIGURE 6. Comparison of the term structures for theoretical and tax-adjusted credit spreads for speculative grade rated bonds.

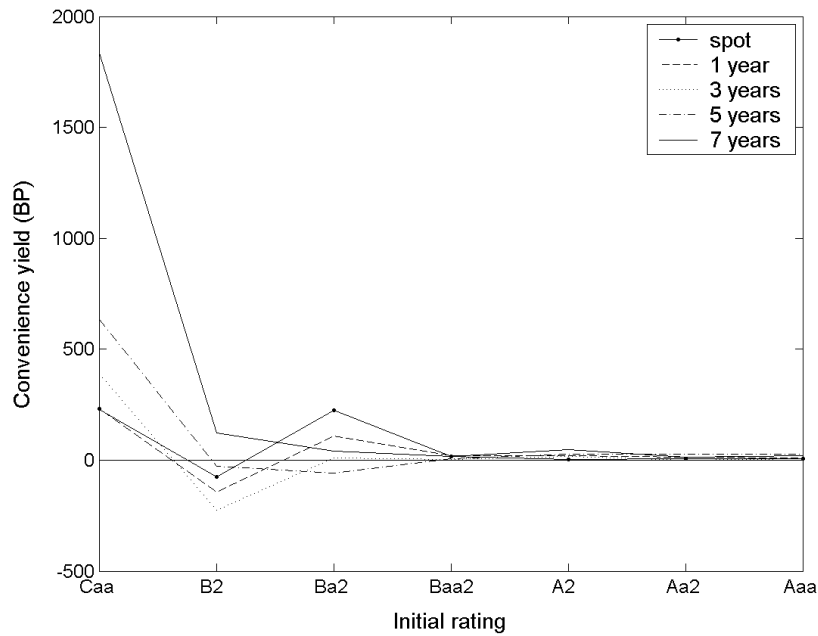


FIGURE 7. Forward liquidity spreads needed to match tax-adjusted market spread rates exactly.

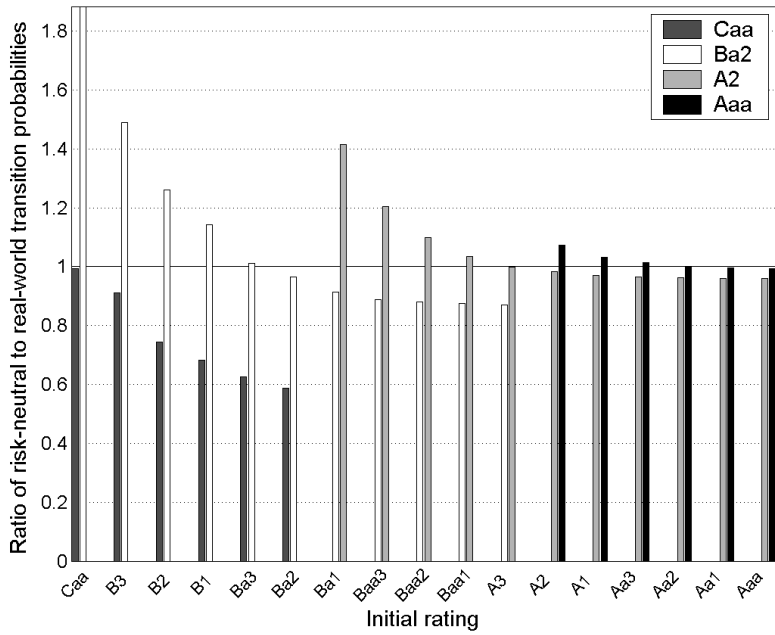


FIGURE 8. Ratio of risk neutral to real world transition probabilities, with a one year time horizon.

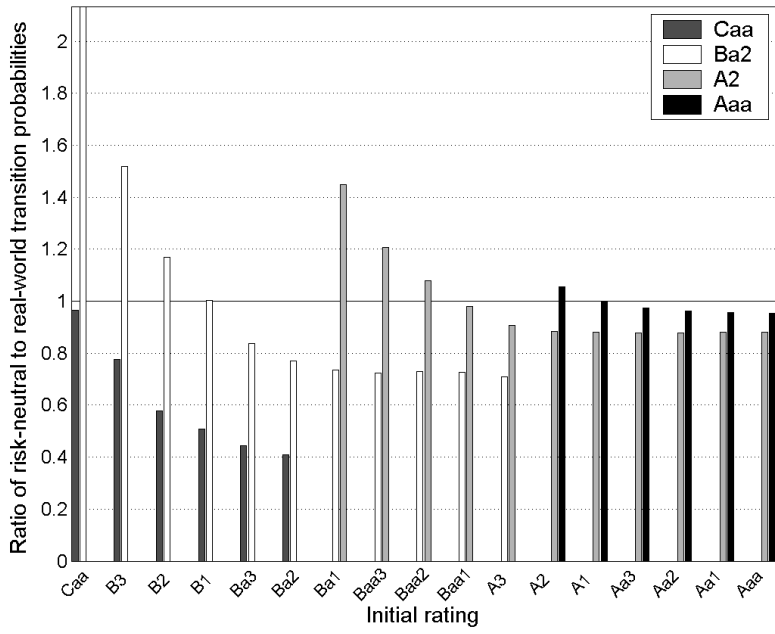


FIGURE 9. Ratio of risk neutral to real world transition probabilities, with a five year time horizon.

## 4. CONCLUSION

We introduced a discrete model for the credit quality process which gives a unified picture of credit migrations and default arrival rates under both the real-world and risk-neutral measures.

Historical credit migration and default rates are reproduced with high accuracy across all rating classes. The process contains an absorbing boundary corresponding to default. The model can also be calibrated to agree with aggregate spread curve data by the introduction of a risk-neutralizing drift that biases the credit process downwards. This gives a consistent model for the aggregate flow of credit quality.

Due to the ability to utilize the Hahn polynomials, calculation of the probability kernels is more efficient than in the continuous case.

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### Appendix. HAHN POLYNOMIALS

We present definitions and properties which we require for the Hahn polynomials. Further results can be found in Nikiforov, Suslov and Uvarov (Springer-Verlag) and Koekoek and Swarttouw (1998).

For  $N$  a positive integer and  $x \in \Lambda = \{0, \dots, N\}$ , the  $n^{\text{th}}$  Hahn polynomial with parameters  $\alpha, \beta > -1$  is defined explicitly by:

$$(A-1) \quad H_n(x; \alpha, \beta, N) = {}_3F_2(-n, n + \alpha + \beta + 1, -x; \alpha + 1, -N|1),$$

where  $0 \leq n \leq N$ . Here,  ${}_3F_2$  is a hypergeometric function defined in general as:

$$(A-2) \quad {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q|z) = \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_p)_j z^j}{(b_1)_j \cdots (b_q)_j j!},$$

where  $(a)_j$  is the Pochhammer symbol, defined as:

$$(a)_j = \begin{cases} a(a+1) \cdots (a+j-1) & j = 1, 2, 3, \dots \\ 1 & j = 0 \end{cases}.$$

Notice that the presence of the  $-n$  and  $-x$  in the first set of parameters of  ${}_3F_2$  in equation (A-1) truncates the infinite sum to a sum containing  $\min(n, x)$  terms. And since  $(-x)_i$  is a polynomial of degree  $i$ , we see that  $H_n(x) = H_n(x; \alpha, \beta, N)$  is a polynomial of degree  $n$  in  $x$ .

Hahn polynomials satisfy the following difference equation

$$(A-3) \quad \lambda_n H_n(x) = B(x)H_n(x+1) - [B(x) + D(x)]H_n(x) + D(x)H_n(x-1)$$

where

$$\begin{aligned} B(x) &= (x + \alpha + 1)(N - x) \\ D(x) &= x(N + \beta + 1 - x) \end{aligned}$$

and

$$(A-4) \quad \lambda_n = -n(n + \alpha + \beta + 1).$$

The following orthogonality relation holds for the Hahn polynomials:

$$(A-5) \quad \sum_{x=0}^N H_m(x)H_n(x)w(x) = \frac{\delta_{m,n}}{d_n^2}.$$

where the measure  $w$  with respect to which they are orthogonal is given by

$$(A-6) \quad w(x) = \binom{\alpha + x}{x} \binom{N + \beta - x}{N - x}$$

and the normalizing factors  $d_n$  are given by:

$$(A-7) \quad d_n^2 = \frac{(-1)^n (2n + \alpha + \beta + 1)(\alpha + 1)_n (-N)_n N!}{(n + \alpha + \beta + 1)_{N+1} (\beta + 1)_n n!}$$

Rather than using the explicit expression for the Hahn polynomials given in (A-1), they can be evaluated recursively by means of the three term recurrence relation

$$(A-8) \quad -xH_n(x) = A_n H_{n+1}(x) - (A_n + C_n)H_n(x) + C_n H_{n-1}(x)$$

where

$$\begin{aligned} A_n &= \frac{(n + \alpha + \beta + 1)(n + \alpha + 1)(N - n)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)} \\ C_n &= \frac{n(n + \alpha + \beta + N + 1)(n + \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 1)}. \end{aligned}$$

The recurrence is initialized with the conditions  $H_{-1}(x) = 0$  and  $H_1(x) = 1$  for all  $x \in \Lambda$ . However, care must be taken in calculating the polynomials computationally as machine errors will propagate exponentially through the recursion. Standard double-precision numbers were insufficient and it was necessary to use arbitrary precision numbers in the calculations to produce the Hahn polynomials. Running on an Intel Pentium 4-1.20GHz, Hahn polynomials for  $N = 1000$  were generated in approximately three minutes.

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