STOCHASTIC MECHANICS AS A GAUGE THEORY

CLAUDIO ALBANESE

Abstract. We introduce a classical diffusion process which provides a full description of non-relativistic Quantum Mechanics and has the form of a $\mathbb{Z}_4$ gauge theory. We first define a stochastic process on a discretization of physical space of the form $(a\mathbb{Z})^3$, where a is an elementary length scale. We then lift this process to the principal bundle $(a\mathbb{Z})^3 \times \mathbb{Z}_4$. Non-relativistic Quantum Mechanics is recovered in the limit as $a \downarrow 0$, as we show in the case of a scalar particle in an electromagnetic field. Many-body interactions can easily be accommodated. In the case of tight binding Hamiltonians no limit needs to be taken, the equivalence is straightforward and sheds new light on the dynamics of quantum phases in solid state Physics. The physical interpretation of Quantum Mechanics in the continuum limit reveals subtle differences between Quantum and Classical Probability and provides an intriguing geometric explanation of quantum coherence and a link to gauge theories.

A question that attracted much attention is whether the Schrodinger equation can be interpreted as a classical diffusion. The first attempt in this direction is in (Fenyes 1952). This was then greatly expanded upon in (Nelson 1967) and became known as Nelson's Stochastic Mechanics. See also (de la Pena-Auerbach 1970), (Jammer 1974), (Guerra and Ruggiero 1973). These efforts were hampered by the difficulty of accommodating many-body interactions while avoiding Bell's inequalities, see (Bell 1966), (Aspect *et al.* 1982). In fact, there appear to be irreconcilable differences between Classical and Quantum Probability. Quantum entanglement shows that different notions of conditioning apply in the two cases.

In this paper, we propose an alternative formulation of Stochastic Mechanics which is very simple and has the form of a gauge theory. It is based on a mathematical calculation revealing that one can exactly embed Quantum Mechanics into a classical diffusion process. The embedding is precise, with no approximations involved, and is discussed in Section 1. This derivation was discovered largely by chance while pursuing an entirely unrelated line of research to obtain numerical rates of convergence for a triangulation scheme for stochastic integrals, see (Albanese 2007). Since some equations in that paper bear some resemblance to Schrodinger equations, it was very tempting to reformat the derivation to emphasize the connection. Guided by this calculation, in Section 2, we give a physical interpretation of this result.

1. From Classical to Quantum Diffusions

Let $a > 0$ be a fixed length scale and consider one particle in the discretized physical space $(a\mathbb{Z})^3$. In the following, we first define a stochastic process in the principal bundle $(a\mathbb{Z})^3 \times \mathbb{Z}_4$. This will suffice in the case of tight binding Hamiltonians. In the continuum, we recover quantum mechanics by taking the limit as $a \downarrow 0$.

Date: First version November 18th, 2007, last revision November 20, 2007.

Let us denote with $\overrightarrow{x} \in (a\mathbb{Z})^3$ a space coordinate and let $n \in \{0,1,2,3\} = \mathbb{Z}_4$. Furthermore, let $\overrightarrow{A}(\overrightarrow{x})$ be a vector potential of an electromagnetic field. Let m and e be the mass and charge of a scalar particle and let c denote the speed of light. We consider a stochastic process on physical space $(a\mathbb{Z})^3$ which can be thought of as the discretization of the diffusion process characterized by the following stochastic differential equation:

$$d\overrightarrow{x}_t = -\frac{2e}{c}\overrightarrow{A}(\overrightarrow{x})dt + \frac{\hbar}{\sqrt{m}}d\overrightarrow{W}. \tag{1.1}$$

More precisely, we consider the process on $(a\mathbb{Z})^3$ of Markov generator

$$\mathcal{L}(\overrightarrow{x};\overrightarrow{x}') = \frac{\hbar^2}{2m}\Delta_a(\overrightarrow{x};\overrightarrow{x}') - \frac{2e}{c}\overrightarrow{A}(\overrightarrow{x})\cdot\overrightarrow{\nabla}_a(\overrightarrow{x};\overrightarrow{x}') \tag{1.2}$$

where

$$\Delta_a(\overrightarrow{x};\overrightarrow{x}') = \sum_{i=1}^{3}\frac{\delta(\overrightarrow{x}'-\overrightarrow{x}-a\overrightarrow{e}_i)+\delta(\overrightarrow{x}'-\overrightarrow{x}+a\overrightarrow{e}_i)-2\delta(\overrightarrow{x}'-\overrightarrow{x})}{a^2} \tag{1.3}$$

$$\nabla_a^i(\overrightarrow{x};\overrightarrow{x}') = \frac{\delta(\overrightarrow{x}'-\overrightarrow{x}-a\overrightarrow{e}_i)-\delta(\overrightarrow{x}'-\overrightarrow{x}+a\overrightarrow{e}_i)}{2a}. \tag{1.4}$$

Let us rearrange this formula as follows:

$$\begin{aligned}\mathcal{L}(\overrightarrow{x};\overrightarrow{x}') = \frac{\hbar^2}{2ma^2}\sum_{i=1}^{3}\Bigg[&\left(1-\frac{4ae}{c}A_i(\overrightarrow{x})\right)\delta(\overrightarrow{x}'-\overrightarrow{x}-a\overrightarrow{e}_i)\\ &+\left(1+\frac{4ae}{c}A_i(\overrightarrow{x})\right)\delta(\overrightarrow{x}'-\overrightarrow{x}+a\overrightarrow{e}_i)-2\delta(\overrightarrow{x}'-\overrightarrow{x})\Bigg].\end{aligned} \tag{1.5}$$

Here, δ is the function such that $\delta(0)=1$ and $\delta(\overrightarrow{x})=0$ if $x \neq 0$. Next, consider the following process on the principal bundle $(a\mathbb{Z})^3 \times \mathbb{Z}_4$:

$$\begin{aligned}\tilde{\mathcal{L}}(\overrightarrow{x},n;\overrightarrow{x}',n') = &\frac{\hbar^2}{2ma^2}\sum_{i=1}^{3}\Bigg[\left(\delta(n'-n+1)-\frac{4ae}{c}\delta(n'-n)A_i(\overrightarrow{x})\right)\delta(\overrightarrow{x}'-\overrightarrow{x}-a\overrightarrow{e}_i)\\ &+\left(\delta(n'-n+1)+\frac{4ae}{c}A_i(\overrightarrow{x})\delta(n'-n)\right)\delta(\overrightarrow{x}'-\overrightarrow{x}+a\overrightarrow{e}_i)-2\delta(\overrightarrow{x}'-\overrightarrow{x})\delta(n'-n)\Bigg]\\ &+\frac{1}{2}V(\overrightarrow{x})\delta(\overrightarrow{x}'-\overrightarrow{x})(\delta(n'-n-1)-\delta(n'-n+1))\\ &+K_0(\delta(n'-n-1)+\delta(n'-n+1)-2\delta(n'-n))\delta(\overrightarrow{x}'-\overrightarrow{x}).\end{aligned} \tag{1.6}$$

Here and in the following, the sum of elements $n \in \mathbb{Z}_4$ is intended to be modulo 4. $\delta(n)=1$ if $n=0$ and zero otherwise. $V(\overrightarrow{x})$ is a potential term including the electrostatic potential and other terms. Further, we assume that

$$|V(x)| \leq 2K_0. \tag{1.7}$$

At the end of the construction, the energy cutoff K_0 will be taken to infinity along with $a \downarrow 0$ to recover quantum mechanics in the limit.

Notice that the lifting above from base space to the principal bundle was defined in such a way that the Markov generator is invariant under translations in the direction of the $\mathbb{Z}_4$ fiber, i.e. up to adding an integer to the n variable, modulo 4. This is a crucial property as it enables us to continue the construction by singling out a sector with respect to this symmetry. Consider the partial Fourier transform operator of kernel

$$\mathcal{F}(\overrightarrow{x},p;\overrightarrow{x}',n) = e^{-ipn}\delta_{xx'} \tag{1.8}$$

where $p = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Since $\mathbb{Z}_4$ translations are a symmetry, partial Fourier transforms are a block-diagonalizing transformation for the lifted generator. Let us introduce the operator $\hat{\mathcal{L}}$ such that

(1.9) $$(\mathcal{F}\tilde{\mathcal{L}}\mathcal{F}^{-1})(\overrightarrow{x},p;\overrightarrow{x}',p') = \hat{\mathcal{L}}(\overrightarrow{x},\overrightarrow{x}';p)\delta_{pp'},$$

i.e.

$$\begin{aligned}\hat{\mathcal{L}}(\overrightarrow{x};\overrightarrow{x}';p) =& \frac{\hbar^2}{2ma^2}\sum_{i=1}^{3}\Bigg[\left(e^{-ip} - \frac{4ae}{c}A_i(\overrightarrow{x})\right)\delta(\overrightarrow{x}' - \overrightarrow{x} - a\overrightarrow{e}_i) \\ &+ \left(e^{-ip} + \frac{4ae}{c}A_i(\overrightarrow{x})\right)\delta(\overrightarrow{x}' - \overrightarrow{x} + a\overrightarrow{e}_i) - 2\delta(\overrightarrow{x}' - \overrightarrow{x})\Bigg] \\ &+ \frac{1}{2}V(\overrightarrow{x})\delta(\overrightarrow{x}' - \overrightarrow{x})(e^{ip} - e^{-ip}) - 2K_0\delta(\overrightarrow{x}' - \overrightarrow{x}).\end{aligned}$$

(1.10) .

Specializing to the sector with $p = \frac{\pi}{2}$, we find

(1.11) $$\hat{\mathcal{L}}\left(\overrightarrow{x};\overrightarrow{x}';\frac{\pi}{2}\right) = i\mathbb{H}(\overrightarrow{x};\overrightarrow{x}') - \left[(1-i)\frac{\hbar^2}{ma^2} + 2K_0\right]\delta(\overrightarrow{x}' - \overrightarrow{x}).$$

where

(1.12) $$\mathbb{H}(\overrightarrow{x};\overrightarrow{x}') = -\frac{\hbar^2}{2m}\Delta_a(\overrightarrow{x};\overrightarrow{x}') + \frac{ie\hbar}{cm}\overrightarrow{A}(\overrightarrow{x})\cdot\overrightarrow{\nabla}_a(\overrightarrow{x};\overrightarrow{x}') + V(x)\delta(\overrightarrow{x}' - \overrightarrow{x}).$$

Let us notice that the kernel of the stochastic process on the principal bundle $(a\mathbb{Z})^3 \times \mathbb{Z}_4$ is given by

(1.13) $$\tilde{u}(t) = e^{t\tilde{\mathcal{L}}} = \mathcal{F}^{-1}\exp\left(t\mathcal{F}\tilde{\mathcal{L}}\mathcal{F}^{-1}\right)\mathcal{F} = \mathcal{F}^{-1}e^{t\hat{\mathcal{L}}}\mathcal{F}.$$

Similarly, we have that

(1.14)

$$\hat{u}\left(\overrightarrow{x},\overrightarrow{x}';\frac{\pi}{2},t\right) = \exp\left(t\hat{\mathcal{L}}\left(\frac{\pi}{2}\right)\right)(\overrightarrow{x},\overrightarrow{x}') = \sum_{n=0}^{3} i^n\tilde{u}(\overrightarrow{x},0;\overrightarrow{x}',n;t) = \exp\left(it\mathbb{H} - (1-i)\frac{\hbar^2 t}{ma^2}\right)(\overrightarrow{x},\overrightarrow{x}')$$

More explicitly, the quantum mechanical kernel can be reconstructed from the probabilistic kernel as follows:

(1.15) $$e^{it\mathbb{H}}(\overrightarrow{x},\overrightarrow{x}') = \exp\left((1-i)\frac{\hbar^2 t}{ma^2} + 2K_0 t\right)\sum_{n=0}^{3} i^n e^{t\tilde{\mathcal{L}}}(\overrightarrow{x},0;\overrightarrow{x}',n;t).$$

This formula shows that the quantum mechanical kernel can be recovered in the limit as $K_0 \to \infty$ and $a \to 0$. In this limit, the exponential factor becomes singular and the phase mixing process infinitely fast, the two effects compensating each other to reproduce precisely standard Quantum Mechanics.

Accommodating several particles is not problematic. A natural approach is to describe each particle with a pair of coordinates $(x,n) \in (a\mathbb{Z})^3 \times \mathbb{Z}_4$. This yields a conserved quantum number for each particle corresponding to phase dynamics. To recover Quantum Mechanics one then averages over the n coordinates for each particle as above.

2. Interpretation

A way of regarding the calculation in Section 1 is as a computational tool which might be useful in situations where explicit modeling of phase dynamics is important. This is particularly straightforward in the case of tight binding Hamiltonians as in the Anderson model, (Anderson 1958), where no limit needs to be taken and the result is just an intriguing mathematical equivalence. A natural application of this representation would be for instance to provide an intuitive representation for electron localization in disordered systems.

In Section 1, we consider only the sector with $p = \frac{\pi}{2}$. The sector with $p = \frac{3\pi}{2}$ also has an interpretation and corresponds to time reversed quantum dynamics. The other two sectors correspond to stochastic processes, one going forward in time and the other going backward.

The latter process is quite singular as the phase dynamics conspires to create resonances. The overall stochastic dynamics though is well defined for all times as long as $a > 0$.

If one takes the limit as $a \to 0$ and $K_0 \to \infty$, our version of stochastic mechanics becomes very close to standard Quantum Mechanics and we notice only formal differences rather than quantitative ones. Firstly, electromagnetism is a $\mathbb{Z}_4$ gauge theory in the case of Stochastic Mechanics and the richer $U(1)$ gauge group of standard Quantum Mechanics emerges only in the limit. Secondly, the very notion of gauge theory is slightly different. Although the gauge group is smaller, the dynamic specification is intertwined with the geometry of the principal bundle. The definition of the lifted dynamics reveals some degrees of freedom and offers opportunities to break parity and time reversal symmetries explicitly. It remains to be investigated how other gauge groups could be incorporated into a similar framework and how one could extend stochastic mechanics to relativistic quantum field theory.

It is also possible to attribute a more fundamental interpretation to this model. The model would be a proper extension of Quantum Mechanics and provide an alternative physical theory if the length scale a was small but finite. This seems a natural assumption in this context. If the elementary length scale is finite, also time can be quantized without spoiling the convergence to Quantum Mechanics. What one would need for the existence of a limit is that time be quantized in such a way that the Courant condition holds

$$\delta t \leq \min_{x,n} \frac{1}{|\tilde{\mathcal{L}}(x,n;x,n)|}. \tag{2.1}$$

The natural appearance of elementary length and time scales in this model would be interesting if the framework could be extended to quantum field theory, as their existence would provide ultraviolet cutoffs. In this context, if the elementary time scale is finite also the constant K_0 would be naturally bounded from above. But perhaps the most intriguing aspect of space-time quantization is the wondrous world at such small distances that our calculation reveals when formulas are interpreted.

The interpretation of the mapping equation (1.15) revolves around the concept that the underlying classical diffusion process we described is not observable in full detail. The relation between the probability density $u(x,n;t)$ of a pure quantum state and a wavefunction $\Psi(x;t)$ is given by

$$\Psi(\overrightarrow{x};t) = N(t) \sum_{n=0}^{3} i^n u(\overrightarrow{x},n;t). \tag{2.2}$$

where $N(t)$ is a normalization constant ensuring that $\sum_x |\Psi(\overrightarrow{x};t)|^2 = 1$. A tenet of Quantum Mechanics is that position and momentum cannot be simultaneously observed. This means that the pair of coordinates x and n cannot be simultaneously observed. The classical diffusion process we wrote describes their dynamics as if they were simultaneously observable and the fact that they actually aren't is modeled through the reinterpretation principle in (2.2).

One possible way to see this is that if there is an elementary time scale $\delta t > 0$, then position cannot be resolved with infinite accuracy. A particle can thus be simultaneously in different locations. This does not infringe upon locality in the limit as $a \to 0$ as delocalization takes place on a fiber $\mathbb{Z}_4$ over a single site $x \in (a\mathbb{Z})^3$.

The impossibility in principle to locate a particle implies that one has to take an average between pairs of locations on the same fiber $\{x\} \times \mathbb{Z}_4$ where it could classically be at a given time. This leads to the consideration of the density

$$\begin{aligned} \Psi(\overrightarrow{x};t)^* \Psi(\overrightarrow{x};t) &= N(t)^2 \sum_{n,m=0}^{3} i^{m-n} u(\overrightarrow{x},n;t) u(\overrightarrow{x},m;t) \\ &= N(t) \sum_{n=0}^{3} \Big[u(\overrightarrow{x},n;t)^2 - u(\overrightarrow{x},n;t) u(\overrightarrow{x},n+2;t) \Big]. \end{aligned} \tag{2.3}$$

An interpretation of this formula can go as follows. The probability distribution function $u(\overrightarrow{x}, n; t)$ gives the probability that an hypothetical classical observer would detect a particle in the state $(\overrightarrow{x}, n)$. However, a physical observer is unable in principle to resolve time to the same degree as the hypothetical (but non-existing) classical observer. As a consequence, any measurement will result in the detection of not one but two particles on two points $(\overrightarrow{x}, n)$ and $(\overrightarrow{x}, n')$ on the same fiber. The variables n and n' can possibly correspond to the same location but are not necessarily equal. One could imaginatively say that due to the fast fiber dynamics, the particle can move during the time the observation is carried out; however one has to be careful with this line of thought as there is more to it. The formula above shows that the probability of detection is given by the product $u(\overrightarrow{x}, n; t)u(\overrightarrow{x}, n'; t)$ i.e. the two locations are treated as independent. Apparently, the formula is telling us that there is no way to condition the probability of detection in the site $(\overrightarrow{x}, n')$ knowing that detection also occurred in the site $(\overrightarrow{x}, n)$. If the time interval is very small, one could say that there could not be such a correlation as one does not know which site was occupied first and causality would prevent us from talking about correlation. This effect is at the origin of quantum coherence.

Another facet of equation (2.3) is that if the particle is detected twice on the same site $(\overrightarrow{x}, n)$ then this counts with the positive sign. If on the other hand a particle is detected in (x, n) and in the antipodal site $(x, n + 2 \bmod 4)$, then this counts with a negative weight. Namely, detection on antipodal sites obfuscates detection. This is the reason why classical probabilities are not observed: they are obfuscated by phase cancelations. If the particle is detected in two different but not antipodal positions, then this does not contribute to the total probability. This gives an interpretation for the quantum mechanical phase cancelation mechanism which results in a systematic loss of classical probability of detection. Since we already established that classical occupation probability is not observable, the apparent loss of classical probability should not come as a surprise. Also, because of this reason, probability of detection must be conditional to detection occurring. Since we don't want to allow for a killing term in the process which would make the particle disappear out of existence, the normalization factor $N(t)$ is required. This emerges also as the exponential function of time in equation (1.15).

The above is not to say that one cannot condition to any event. Classical probability conditioning still applies in many situations except that it does not apply in the particular case of the detection of position within a single fiber because, given the shortness of the time interval, conditioning in this case would break causality. If the quantum system is in the presence of a classical environment, classical probabilistic conditioning can occur depending on events whose classical probabilities (P_k) are known, where $k = 0, 1, ...$ and $\sum_k P_k = 1$. In this case, the status of the quantum system is represented by a density matrix

$$\rho(x; x') = \sum_k P_k \Psi_k(x)^* \Psi_k(x') \tag{2.4}$$

The particle density is given by the diagonal elements

$$\rho(x; x) = \sum_k \Psi_k(x)^* \Psi_k(x). = \sum_{n=0}^{3} \left[\sum_k P_k N_k(t) u(\overrightarrow{x}, n; t|k)^2 - \sum_k P_k N_k(t) u(\overrightarrow{x}, n; t) u(\overrightarrow{x}, n+2; t|k) \right]. \tag{2.5}$$

Hence, the probability of simultaneous detection in $(\overrightarrow{x}, n)$ and $(\overrightarrow{x}, n)$ is the sum of conditional probabilities over all the classical conditioning events.

As much as this interpretation of short distance physics on quantized space-time will undoubtedly appear counter-intuitive, there is logic to it. The mathematical reduction of a classical diffusion to a quantum one in this paper is just a derivation and the physical interpretation is a way of interpreting equations. This framework gives a Rosetta stone for reinterpreting geometrically other quantum oddities and concepts that are difficult to fully grasp such as quantum entanglement and co-existence of classical and quantum objects in the same equations. The reader accustomed to Quantum Mechanics may find the key differences between classical and quantum probabilities and the notion of quantum coherence perhaps even easier to conceptualize if expressed in this language. This would be similar to transliterating from hieroglyph to

English without changing the content of much. Whether this implies that space-time is actually quantized or that the elementary scales we introduce are in any way related to the Planck scale or even quantum gravity, will be a point to be left entirely untouched.

References

Albanese, C. (2007). Stochastic Integrals and Abelian Processes. *arXiv:0711.2980 [math.NA]*.
Anderson, P. W. (1958). Absence of Diffusion in Certain Random Lattices. *Phys.Rev* **109**, 1492–1505.
Aspect, A., J. Dalibard and G. Roger (1982). *Phys. Rev. Letters* **49**, 1804–1807.
Bell, J.S. (1966). *Rev. Mod. Phys.* **38**, 447.
de la Pena-Auerbach, L. (1970). *Phys. Letters* **31A**, 403404.
Fenyes, I. (1952). *Z.Physik* **132**, 81–106.
Guerra, F. and P. Ruggiero (1973). *Phys. Rev. Letters* **31**, 1022.
Jammer, M. (1974). *The Philosophy of Quantum Mechanics*. Wiley.
Nelson, E. (1967). *Dynamical Theories of Brownian Motion*. Princeton University Press.

E-mail address: claudio@level3finance.com