A STOCHASTIC VOLATILITY MODEL FOR BERMUDA SWAPTIONS AND CALLABLE CMS SWAPS

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ABSTRACT. It is widely recognized that fixed income exotics should be priced by means of a stochastic volatility model. Callable constant maturity swaps (CMS) are a particularly interesting case due to the sensitivity of swap rates to implied swaption volatilities for very deep out of the money strikes. In this paper, we introduce a stochastic volatility term structure model based on a continuous time lattice which allows for a numerically stable and quite efficient methodology to price fixed income exotics in this class.

1. Introduction

The history of interest rate models is characterized by a long series of turns. The Black formula for caplets and swaptions was designed to take as underlying a single forward rate under the appropriate forward measure. This approach has the advantage to lead to a simple pricing formula for European options but also the limitation of not being extendable to callable contracts. To have a more consistent model, short rate models where introduced in (Cox, Ingersoll & Ross 1985), (Vasicek 1977), (Black & Karasinski 1991) and (Hull & White 1993). Next came LIBOR market models, also known as correlation models. First introduced in (Brace, Gatarek & Musiela 1996) and (Jamshidian 1997), forward LIBOR models affirmed themselves as a flexible Montecarlo pricing methodology providing non-parametric fits to both term structures for interest rates and at-the-money Black volatilities for either caplets or a family of swaptions of fixed tenor. Various extensions of forward LIBOR models aim at incorporating volatility smiles, calibrating thus to the so called "volatility cube", given by the Black volatility of swaptions as a function of strike, maturity and tenor. Local volatility extensions were pioneered in (Andersen & Andreasen 2000). A stochastic volatility variation is proposed in (Andersen & Brotherton-Ratcliffe 2001), and is further extended in (Andersen & Andreasen 2002). A different approach to stochastic volatility forward LIBOR models is described in (Joshi & Rebonato 2003). Jump-diffusion forward LIBOR models are treated in (Glasserman & Merener 2001), (Glasserman & Kou 1999). A calibration framework is proposed in (Piterbarg 2003).

Modeling stochastic volatility within the framework of LIBOR market models is a challenging task from an implementation viewpoint due to the intrinsic simulation noise of Montecarlo methods which make the calculation of hedge ratios a particularly arduous task. In a strive to achieve a better understanding of the volatility dynamics and vega sensitivities, in recent years we witnessed a move away from correlation models and the emergence and recognition as a market standard in the fixed income domain of the SABR model by (S.Hagan, Kumar, S.Lesniewski & E.Woodward 2002). Although probably not the ultimate solution, SABR is an important

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stepping stone which allows one to better understand the stochastic volatility process from an analytical viewpoint. Likewise to the Black formula approach, SABR models as underlier a single forward rate under the corresponding forward measure. A limitation of this framework is that it is not suitable to price callable swaps and Bermuda swaptions. Pricing inconsistencies may also emerge with SABR due to the fact that the model is solved by means of expansions which are asymptotic (as opposed to convergent) and are thus characterized by irreducible approximation errors.

Within the standard SABR one also observes calibration difficulties with European options. Implied swaption volatilities with very large strikes are probed by constant maturity swaps (CMS). These exotic structures receive a floater given by a spread over LIBOR and pay the equilibrium swap rate of a fixed tenor prevailing at each coupon date. An analysis of the coupon structure leads to the conclusion that CMS contracts are particularly sensitive to the asymptotic behavior of implied volatilities for very large strikes. Market CMS rates actually drive the option market in extreme strike regions and indicate that implied volatilities flatten out and converge asymptotically to a constant. This behavior is not consistent with the rapidly diverging asymptotics which are implied by SABR.

In this article, we attempt to go beyond SABR by introducing a stochastic volatility short rate model which has the correct asymptotic behavior for implied swaption volatilities and can thus be used for callable swaps and Bermuda swaptions. Our model is solved on continuous time lattices of a new type. Discretization schemes of similar type have been previously discussed in (Albanese & Kuznetsov 2005) and (Albanese & Kuznetsov 2003) and applied to models for which the spectrum of the Markov generator can be computed in analytically closed form. While retaining the insights provided by the spectral analysis treatment in the previous papers, here we renounce to analytic solvability and follow instead a non-parametric approach. The new idea is to leverage not on the ability to evaluate special functions but instead on numerical linear algebra routines. In particular, no part of our calculations require the use of Montecarlo or asymptotic methods and prices and hedge ratios are very stable, even for extreme strike levels.

This technique yields a gain in model flexibility and allows us to extend the application domain and refine the calibration procedure. The criterion guiding our methodology is time-stationarity and calibration is carried out in three separate stages: in the first we achieve an approximate fit to the term structure of interest rates and the term structure of implied at-the-money volatilities for a selected family of hedging vehicles. In the second we make the fit to interest rates and implied at-the-money volatility by introducing a minimum amount of time dependence through a time-dependent coordinate change. This provides a very reasonable but not completely accurate fit to out-of-the-money implied volatilities. This issue is addressed in a third calibration step aimed at achieving perfect consistency with out-of-the-money skews in the region of the volatility cube spanned by our selected hedging vehicles. For this purpose, we make use of a sequence of measure changes which resemble a technique introduced in the context of the Markov functional model, see (Hunt, Kennedy & Pelsser 2000), except that we work consistently in the risk neutral measure as opposed a terminal bond as numeraire asset. As a result of the construction, our model is nearly stationary over time horizons in excess of 30 years.

In this article, we present the model by reviewing in detail an implementation example. While leaving it up to the interested reader to pursue the many conceivable variations and extensions, we describe the salient features of our modeling by focusing in detail to the problem of pricing and finding hedge ratios for Bermuda swaptions and callable CMS swaps.

2. The model

Our model is built upon a specification of a short rate process r_t which combines local volatility, stochastic volatility and jumps. The calibration procedure consists of three stages. In the first, we make our best effort to calibrate the model to a stationary process. In the second, we introduce the least possible degree of explicit time dependence in such a way to refine fits of the term structure of interest rates, of selected at-the-money swaption volatilites and of the asymptotic behaviour of the implied swaption smiles. A series of measure changes is finally overlayed on top of this model in such a way to achieve full consistency with the vol-cube data implied by European style CMSs. The model is specified in a largely non-parametric fashion within a functional analysis formalism and expressed in terms of continuous time lattice models.

A sequence of several steps is required to specify the short rate process r_t.

2.1. **The conditional local volatility processes.** We introduce M states of volatility. The process conditioned to stay in one of such states $\alpha \in \{1, ...M\}$ is related to the solution $r_{\alpha t}$ of the following equation:

$$dr_{\alpha t} = \kappa_\alpha(\theta_\alpha - r_{\alpha t})dt + \sigma_\alpha r_{\alpha t}^{\beta_\alpha} dW. \tag{1}$$

In the functional analysis formalism we use, these SDEs are associated to the Markov generators

$$\mathcal{L}_\alpha^r = \kappa_\alpha(\theta_\alpha - r_{\alpha t})\frac{\partial}{\partial r} + \frac{\sigma_\alpha^2 r_{\alpha t}^{2\beta_\alpha}}{2}\frac{\partial^2}{\partial r^2}. \tag{2}$$

To build a continuous time lattice, we discretize the short rate variable and constrain it to belong to a finite lattice Ω containing $N+1$ points $r(x) \geq 0$, where $x = 0, ...N$, $r(0) = 0$ and the following ones are in increasing order, not necessarily equally spaced. The discretized Markov generator $\mathcal{L}_{\Omega\alpha}^r$ is defined as the operator represented by a tridiagonal matrix such that

$$\begin{aligned}
\sum_y \mathcal{L}_{\Omega\alpha}^r(x, y) &= 0 \\
\sum_y \mathcal{L}_{\Omega\alpha}^r(x, y)(y - x) &= \kappa_\alpha(\theta_\alpha - r(x)) \\
\sum_y \mathcal{L}_{\Omega\alpha}^r(x, y)(y - x)^2 &= \sigma_\alpha^2 r(x)^{2\beta_\alpha}.
\end{aligned}$$

In our example, we select an inhomogenous grid of $N = 70$ points spanning short rates from 0% to 50%, in such a way to obtain a finer grid in the region around the spot rate level. We also choose to work with $M = 4$ states for volatility and make the following parameter choices (see Fig. 1):

α	σ_α	β_α	θ_α	κ_α
0	31%	30%	2.10%	.17
1	46%	40%	5.50%	.18
2	75%	50%	8.50%	.23
3	100%	60%	12.00%	.24

Although our model is more complex than a simple local volatility process, it is convenient to describe our resolution method in the specific case of the operator $\mathcal{L}_{\Omega\alpha}^r$ with constant α. This method can then be generalized and is at the basis of other extensions such as the introduction of jumps (see below). We start by considering the following pair of eigenvalue problems:

$$\mathcal{L}_{\Omega\alpha}^r u_n = \lambda_n u_n \qquad\qquad \mathcal{L}_{\Omega\alpha}^{rT} v_n = \lambda_n v_n$$

where the superscript ^{T} denotes matrix transposition, u_n and v_n are the right and left eigenvectors of $\mathcal{L}^r_{\Omega\alpha}$, respectively, whereas λ_n are the corresponding eigenvalues. Except for the simplest cases, the Markov generator $\mathcal{L}^r_{\Omega\alpha}$ is not a symmetric matrix, hence u_n and v_n are different. Also, in general the eigenvalues are not real. We are only guaranteed that their real part is non-positive $\mathcal{R}e\lambda_n \leq 0$ and that complex eigenvalues occur in complex conjugate pairs, in the sense that if λ_n is an eigenvalue then its conjugate $\bar{\lambda_n}$ is also an eigenvalue. We set boundary conditions in such a way that there is absorption at the endpoints, and in particular at the point corresponding to zero rates $r(0) = 0$. With this choice, we guarantee probability conservation for the process, and as a consequence there exists a zero eigenvalue.

There is no guarantee, in the most general case, that there exists a complete set of eigenfunctions. However, the chance that such a complete set does not exist for a Markov generator specified non-parametrically is zero, so we can safely assume that this is the case. In the unlikely case that this assumption is not correct, the numerical linear algebra routines needed to solve our lattice model will identify the problem and a small perturbation of the given operator will suffice to rectify the situation. Assuming completeness, the diagonalization problem can be rewritten in the following matrix form:

$$\mathcal{L}^r_{\Omega\alpha} = U\Lambda U^{-1} \tag{3}$$

where U is the matrix having as columns the right eigenvectors and Λ is the diagonal matrix having the eigenvalues λ_i as elements.

Key to our constructions is the remark that, if the Markov generator is diagonalisable, we can apply an arbitrary function F to it by means of the following formula:

$$F(\mathcal{L}^r_{\Omega\alpha}) = UF(\mathcal{L}^r_{\Omega\alpha})U^{-1} \tag{4}$$

This formula is at the basis of the so-called "functional calculus". As Ito's formula regarding functions of stochastic processes is central in the mathematical Finance for diffusion processes, functional calculus for Markov generators plays a pivotal role in our framework for stochastic volatility models. This formula has several applications. An immediate one allows us to express the pricing kernel $u(r(x), t; r(y), T)$ of the process as follows:

$$u(r(x), t; r(y), T) = (e^{(T-t)\mathcal{L}^r_{\Omega\alpha}})(x, y) = \sum_n e^{\lambda_n (T-t)} u_n(x)v_n(y). \tag{5}$$

2.2. **Introducing jumps.** At this stage of the construction one has the option to also add jumps. Although in the example discussed in this paper we are mostly focused on long dated callable swaps and swaptions for which we find that the impact of jumps can be safely ignored, adding jumps involves negligible additional complexities and is thus worth considering and implementing in other situations. To add jumps, one can follow the following procedure which accounts for the need to assign different intensities to up-jumps and down-jumps. Jump processes are associated to a special class of stochastic time changes given by monotonously non-decreasing processes T_t with independent increments. These time changes are known as Bochner subordinators and are characterized by a Bochner function $\phi(\lambda)$ such that

$$E_0\left[e^{-\lambda T_t}\right] = e^{-\phi(\lambda)t} \tag{6}$$

For example, the case of the variance gamma process which received much attention in Finance corresponds to the function

$$\phi(\lambda) = \frac{\mu^2}{\nu}\log\left(1 + \lambda\frac{\nu}{\mu}\right) \tag{7}$$

where μ is the mean rate and ν is the variance rate of the variance gamma process. The generator of the jump process can be expressed using functional calculus as the operator $-\phi(-\mathcal{L}^r_{\Omega\alpha})$. To produce asymmetric jumps, we specify the two parameters in (7) differently for the up and down jumps and compute separately two Markov generators

$$\mathcal{L}_{\pm} = -\phi_{\pm}(-\mathcal{L}^r_{\Omega\alpha}) = -U_{\pm}\phi_{\pm}(\Lambda)U^{-1}{}_{\pm} \tag{8}$$

where:

$$\phi_{\pm}(\lambda) = \frac{\mu_{\pm}^2}{\nu_{\pm}} \log(1 + \lambda\frac{\nu_{\pm}}{\mu_{\pm}}) \tag{9}$$

The new generator for our process with asymmetric jumps is obtained by combining the two generators above

$$\mathcal{L}^r_{j\Omega\alpha} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \mathcal{L}_-(2,1) & d(2,2) & \mathcal{L}_+(2,3) & \cdots & \mathcal{L}_+(2,n) \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \mathcal{L}_-(n-1,1) & \mathcal{L}_-(n-1,2) & \cdots & d(n-1,n-1) & \mathcal{L}_+(n-1,n) \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

Here the element of the diagonal are chosen in such a way to satisfy probability conservation:

$$d(x,x) = -\sum_{y \neq x} \mathcal{L}^r_{j\Omega\alpha}(x,y) \tag{10}$$

Also notice that we have zeroed out the elements in the matrix at the upper and lower boundary: this ensures that there is no probability leakage in the process.

In our setting, we choose a short rate as a modeling primitive and we thus do not need to impose a martingale condition. Otherwise, were we working with a forward rate instead, the appropriate method of restoring the martingale condition would be to modify the matrix elements of the resulting generator on the first sub-diagonal and the first super-diagonal.

At this stage of the construction, we have therefore obtained a generator $\mathcal{L}^r_{j\Omega\alpha}$ for the short rate process, whose dynamics is characterized by a combination of state dependent local volatility and asymmetric jumps. We note that the addition of jumps has not increased the dimensionality of the problem and is therefore computationally efficient.

2.3. **Modeling the dynamics of stochastic volatility.** Next we define a dynamics for stochastic volatility by assigning a Markov generator to the volatility state variable α which depends on the rate coordinate x. Namely, conditioned to the rate variable being x, the generator has the following form

$$\mathcal{L}^{sv}_x = \epsilon(x)\mathcal{L}^{sv}_+ + \mathcal{L}^{sv}_- \tag{11}$$

where the excitability function $\epsilon(x)$ is graphed in Fig. 2 and

$$\mathcal{L}^{sv}_+ = \begin{pmatrix} -0.7 & 0.7 & 0 & 0 \\ 0 & -1.1 & 0.8 & 0.3 \\ 0 & 0 & -1.5 & 1.5 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{L}^{sv}_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1.4 & -1.4 & 0 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 5 & -5 \end{pmatrix}. \tag{12}$$

This construction allows to describe an asymmetric behavior of the stochastic volatility process: the relaxations, from a regime of higher volatility to one of lower volatility, are defined by the operator $\mathcal{L}^{sv}_-$, whilst the excitations, from a regime of lower volatility to one of higher volatility, are defined by the operator $\mathcal{L}^{sv}_+$. Also, state dependency in the excitations is introduced through the function $\epsilon(x)$, in order to dampen the stochastic volatility process at low values of the rate coordinate x. The definition of the operator $\mathcal{L}^{sv}_x$ plays an important role in the calibration of the model to the implied volatility smiles obtained from market observations, and it is crucial in order to obtain the correct asymptotic behavior of implied volatilities at very large strikes.

Out of the two generators we just defined, we form a Markov generator $\mathcal{L}$ acting on functions of both the rate variable x and the volatility variable α. This generator has matrix elements given as follows:

$$\mathcal{L}(x, \alpha; y, \beta) = \mathcal{L}^r_{j\Omega\alpha}(x, y)\delta_{\alpha\beta} + \mathcal{L}^{sv}_x(\alpha, \beta)\delta_{xy}. \tag{13}$$

where δ indicates the Kronecker delta.

To gain an intuition of the meaning of the matrix element $\mathcal{L}(x, \alpha; y, \beta)$ we note that, for each element off the main diagonal, $\mathcal{L}(x, \alpha; y, \beta)dt$ represents the transition probability to move from a state $r(x)$ in the α volatility regime to a state $r(y)$ in the β volatility regime. We are therefore describing a two-dimensional problem: one for the underlying short rate and one for the stochastic volatility.

In our working example, the matrix $\mathcal{L}$ has total dimension $MN = 280$. For matrices of this size, diagonalization routines such as `dgeev` in LAPACK are very efficient. Since our underlier is a short rate though, we are not interested in the pricing kernel but rather in the discounted transition probabilities given by

$$G(x, t; y, T) = E\left[e^{-\int_t^T r_s ds}, |r_t = r(x), r_T = r(y)\right]. \tag{14}$$

This kernel satisfies the following backward equation

$$\frac{\partial}{\partial t} G(x, t; y, T) + (\mathcal{L}G)(x, t; y, T) = r(x)G(x, t; y, T). \tag{15}$$

In functional calculus notations, the solution is given by

$$G(x, t; y, T) = e^{\mathcal{G}(T-t)}(x, y) \qquad \text{where} \qquad \mathcal{G}(x, y) \equiv \mathcal{L}(x, y) - r(x)\delta_{xy}. \tag{16}$$

The same diagonalization method illustrated above for the local volatility case applies also in this situation. By representing the matrix $\mathcal{G}$ in the form

$$\mathcal{G} = U\Lambda U^{-1} \tag{17}$$

where Λ is diagonal, and writing the matrix of discounted transition probabilities as follows

$$e^{\mathcal{G}(T-t)} = Ue^{\Lambda(T-t)}U^{-1}. \tag{18}$$

2.4. **Measure changes.** At the third and final calibration stage, we overlay a series of measure changes to fine-tune the value of out-of-the-money implied volatility skews up to fairly remote strike regions. This introduces a degree of time dependence in the model which however is fairly limited, due to the fact that the first calibration step typically produces a very reasonable first approximation with the correct large strike asymptotic behavior, as we discuss below.

Our sequence of measure changes does not correspond to changes of numeraire asset, as is typically done with interest rate derivatives, see (Jamshidian 1993) and (Geman, Karoui & Rochet 1995), but is rather of a slightly different nature, as we work exclusively under the risk neutral measure. Consider an increasing sequence of maturities for European style options for which we wish to obtain a perfect price fit: $t_1, t_2, ..., t_n$. The desired measure change is specified by a sequence of functions $H_i(x) > 0$ for $i = 1, \ldots, n$ such that, for $t \,\epsilon\, (t_{i-1}, t_i)$ we have that

$$\frac{\partial}{\partial t} H_i(x, t; y, t_i) + (\mathcal{L}H_i)(x, t; y, t_i) = 0. \tag{19}$$

For each $t \,\epsilon\, (t_{i-1}, t_i)$, the measure-changed discounted transition probabilities are given by

$$G_{H_i}(x, t; y, t_i) = \frac{1}{H_i(x, t)} G(x, t; y, t_i) H_i(y, t_i) \tag{20}$$

and they satisfy to the following backward equation:

$$\frac{\partial}{\partial t} G_{H_i}(x, t; y, t_i) + ((\mathcal{L}_{H_i} - r) G_{H_i})(x, t; y, t_i) = 0. \tag{21}$$

where

$$(\mathcal{L}_{H_i} - r)(x, t; y, t_i) = \frac{1}{H_i(x, t)} (\mathcal{L} - r)(x, t; y, t_i) H_i(y, t_i) + \frac{1}{H_i(x, t)} \frac{\partial}{\partial t} H_i(x, t). \tag{22}$$

Composing the measure-changed process across time intervals, the measure changed kernel for the discounted transition probabilities, for $0 \le i < j \le n$, is given by

$$G_H(x_i, t_i; x_j, t_j) = \sum_{x_{i+1}, \ldots, x_{j-1}} \prod_{k=i+1}^{j} \frac{H_k(x_k, t_k)}{H_k(x_{k-1}, t_{k-1})} G(x_{k-1}, t_{k-1}; x_k, t_k) \tag{23}$$

where H_k is the measure change function defined at time t_k.

The measure change functions H_k provide the time dependent degree of freedom we need to calibrate to the region of the vol-cube spanned by our selected hedging vehicles. The defined measure-change will depend on the particular family of hedging vehicles we select. The fit to the other regions of the vol cube, comprising swaptions of different tenor and maturities is qualitatively correct but not quantitatively precise. Further reducing these discrepancy will require augmenting the number of regimes and refining the model. This would prove useful to extend our model to other exotics such as for instance CMS spreads. We currently have work in progress in this direction and plan to return on this subject with a more systematic treatment in a follow up article.

3. Calibration and Pricing

In our example, to calibrate our model we aim at matching forward swap rates and at-the-money swaption volatilities, both referring to swaps of 5 year tenor. We consider here in detail the case of EUR denominated swaptions and use for comparison purposes also JPY denominated swaptions. We make use of datasets referring to market quotes in July 2005. The EUR data is given in the following table:

	$1y$	$2y$	$3y$	$4y$	$5y$	$7y$	$10y$	$15y$	$20y$	$30y$
forward	2.999%	3.311%	3.587%	3.800%	3.984%	4.226%	4.393%	4.477%	4.301%	4.114%
ATM vol	21.506%	19.443%	17.962%	16.967%	16.189%	14.897%	13.801%	12.460%	12.665%	11.728%

In a first calibration step, we search for a best fit using the model above without introducing any explicit time dependency. In a second step, we then introduce time dependency to achieve a perfect fit to the term structure of forward rates and at-the-money swaption volatilities. As a consequence of this procedure, the degree of time variability of model parameters is kept to a bare minimum. As we calibrate the model to at-the-money Black volatilities, we find that the high degree of stationarity ensures a qualitatively correct asymptotic behavior for implied swaption volatilities even in the region of extreme strike values. Fig. 5 and Fig. 6 show our results for EUR and JPY denominated swaptions. This qualitative agreement also ensures that the subsequent series of measure changes adds only a small amount of explicit time dependence.

To fit term structures of interest rates and implied volatilities, we introduce explicit time dependence by combining two operations: a shift of the short rate by a time varying, deterministic function of time and a deterministic time change, i.e.

$$r_t \to \tilde{r}_t = \dot{b}(t) r_{b(t)} + a(t). \tag{24}$$

Here $b(t)$ is monotonously increasing and $\dot{b}(t)$ denotes its time derivative. Using the new process, discounted transition probabilities can be computed as follows:

$$E\left[e^{-\int_t^T \tilde{r}_s ds}, |\tilde{r}_t = \tilde{r}(x), \tilde{r}_T = \tilde{r}(y)\right] = e^{-\int_t^T a(s)ds} G(x, b(t); y, b(T)) \tag{25}$$

where G is the kernel for the stationary process defined above.

Our choice in the working example is $b(t) = 1.095t + 0.008t^2$. The function $a(t)$ is then defined in such a way to match the term structure of forward swap rates. This adjustment is given in Fig. 3. As one can see from this picture, the short rate adjustment is typically less than 10 basis points in absolute value. This ensures that the probability of the modified short rate process $\tilde{r}_t$ to attain negative values is very small. In contrast, in a typical implementation of the Hull-White model along similar lines, the short rate adjustment is typically of a few percent. The discrepancy is linked to the fact that the richer stochastic volatility model we construct is capable of explaining most of the salient features of the zero curve even with the constraint that the process be stationary.

An additional advantage of having a nearly stationary model is that the shapes of yield curves that one obtains depend on the short rate and the volatility state but are largely independent of time. Fig. 4 shows the yield curves corresponding to different initial volatility states and different starting values for the short rate. As the graphs indicate, yield curves are sensitive to the initial volatility state as they raise if initial volatilities raise. Moreover graphs show that curves invert for high values of the short rate. In our model, this behavior is consistent over all time frames except for corrections of the order of 10 basis points.

3.1. **Pricing swaptions and callable constant maturity swaps.** We apply our model to the pricing of Bermuda swaption and callable CMS swaps. The latter are of particular interest, as the pricing of these contracts is very sensitive to the asymptotic behaviour of the implied volatility smiles of the model. Our numerical results refer to EUR denominated CMSs. We find that the model reproduces the correct asymptotic behaviour of the implied volatility smiles and gives rise to numerically stable exercise boundaries and hedge ratios.

Implied volatilities for European swaptions are given in Fig. 5 and 6. Here we graph extreme out of the money strikes of up to 15% for swaptions of varying maturities where the deliverable is a 5Y swap. One can notice that implied volatilities naturally flatten out at long maturities, a behavior consistent with what observed in the european swaption market and also inferred from the CMS market where such extreme strike levels are probed. Adopting this as a starting point, a perfect non-parametric fit can be achieved using the sequence of measure changes described above, without introducing a significant amount of explicit time dependence.

Exercise boundaries for 10Y Bermuda swaptions are given in Fig. 7 and Fig. 8. The first graph refers to payer swaptions and the second to receiver swaptions. The corresponding graphs for callable CMSs are given in Fig. 9 and Fig. 10. Notice that the exercise boundaries depend on the volatility state.

Sensitivities for Bermuda swaptions are given in Fig. 11 and 12. These sensitivities are computed by holding the volatility state variable fixed and are defined as the derivative of the price for a 10Y payer Bermuda swaption, struck at a the rate reported on the abscissa, with respect to the rate of the 10Y swap. Numerically, these sensitivities are obtained by moving the spot rate and computing the change in value of both the Bermuda swaption contract and the underlying hedging instrument.

Sensitivities of callable constant maturity swaps are given in Fig. 13, 14 and 15. The delta and gamma are computed similarly to what done for Bermuda swaptions, while the vega is calculated with respect to the 10Y into 5Y European swaption.

4. Conclusions

We present a stochastic volatility term structure model which provides a consistent framework for pricing European and Bermuda options, as well as callable CMS swaps. The model is built upon a specification of a short rate process, which combines local volatility, stochastic volatility and jumps. The richness of the model allows to keep the degree of time variability of model parameters to a bare minimum, and obtain a nearly stationary dynamic, while achieving also a perfect fit to the term structure of interest rates and a region of the volatility cube corresponding to a selected family of hedging vehicles.

The solution methodology is based on a new type of continuous time lattices, which allow for a numerically stable and quite efficient technique to price fixed income exotics and evaluate hedge ratios. The mathematical technique here described is quite flexible and is perhaps the main contribution of this paper.

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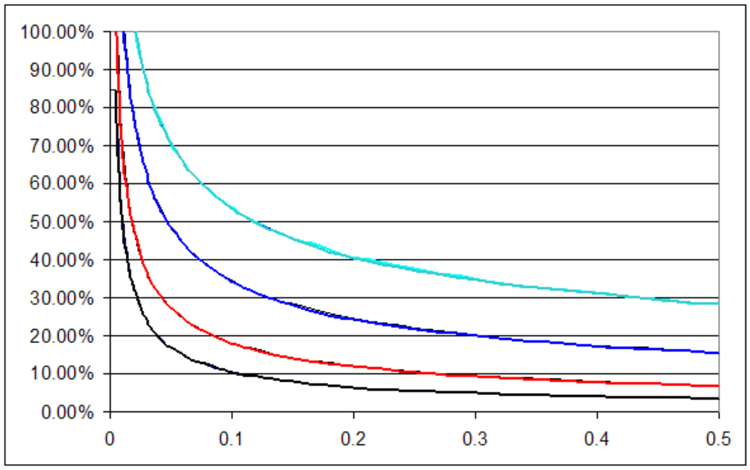

FIGURE 1. Short rate volatilities for each of the four volatility states

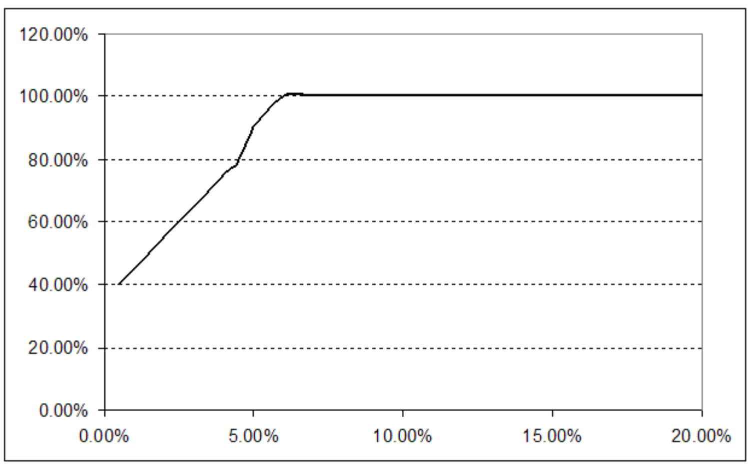

FIGURE 2. Excitability function $\epsilon(x)$

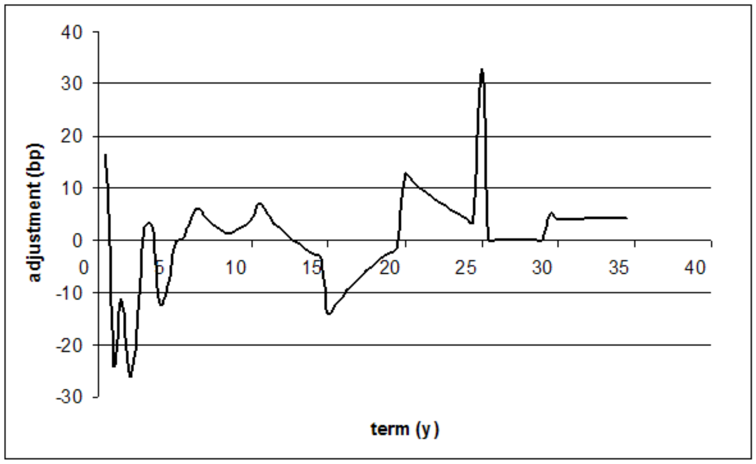

FIGURE 3. Deterministic short rate adjustment

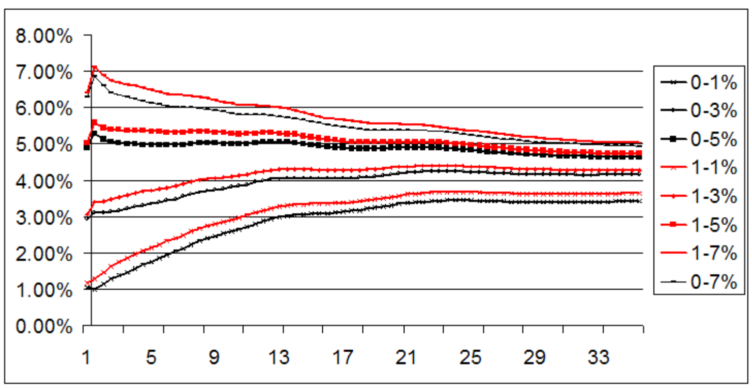

FIGURE 4. Yield curves for different values of the initial volatility state and of the short rate

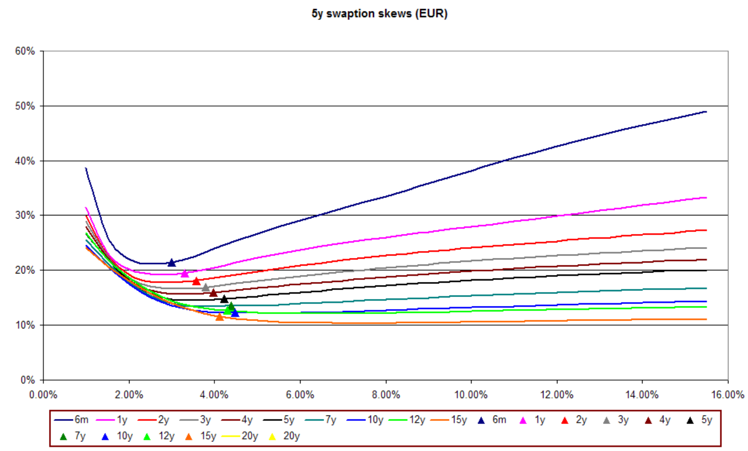

FIGURE 5. Implied volatility for 5-year EUR denominated European swaptions

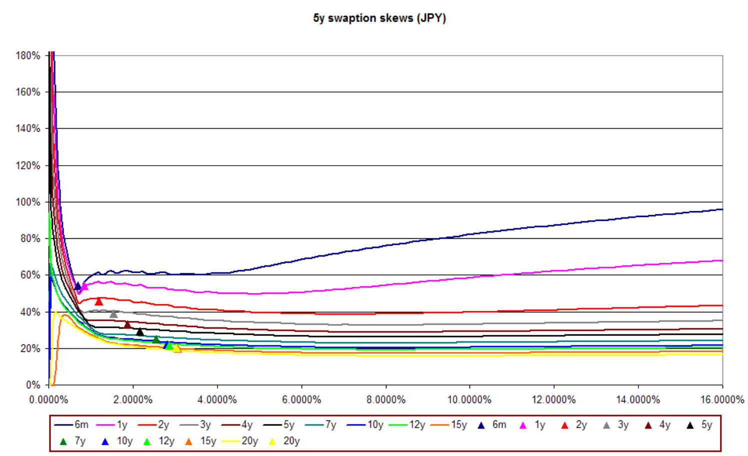

FIGURE 6. Implied volatility for 5-year JPY denominated European swaptions

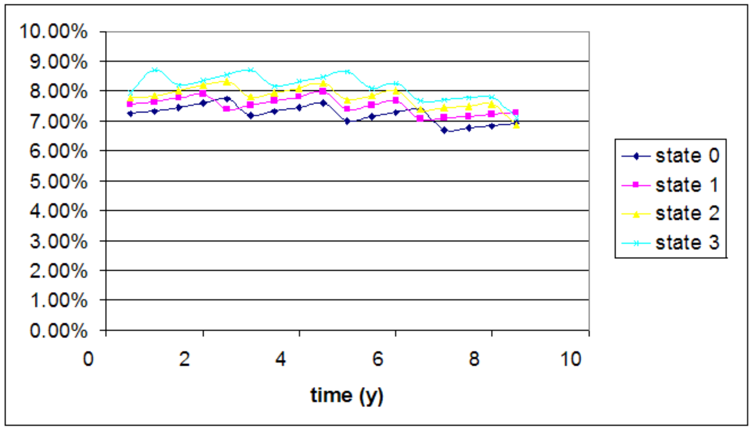

FIGURE 7. Exercise boundaries for payer Bermuda options (EUR denominated)

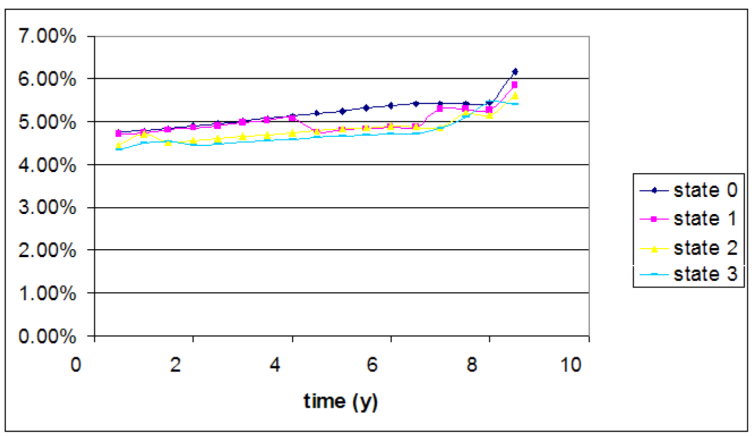

FIGURE 8. Exercise boundaries for receiver Bermuda options (EUR denominated)

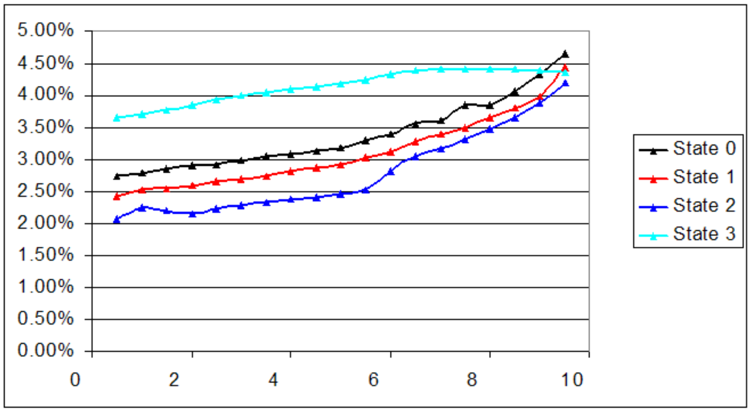

FIGURE 9. Exercise boundaries for callable payer CMSs (EUR denominated)

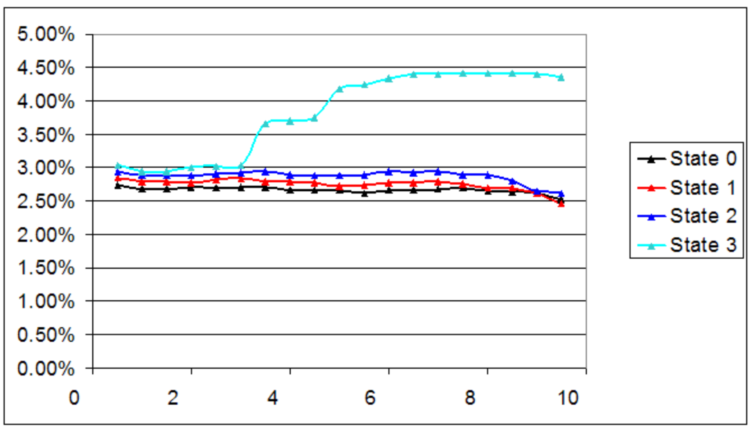

FIGURE 10. Exercise boundaries for callable receiver CMSs (EUR denominated)

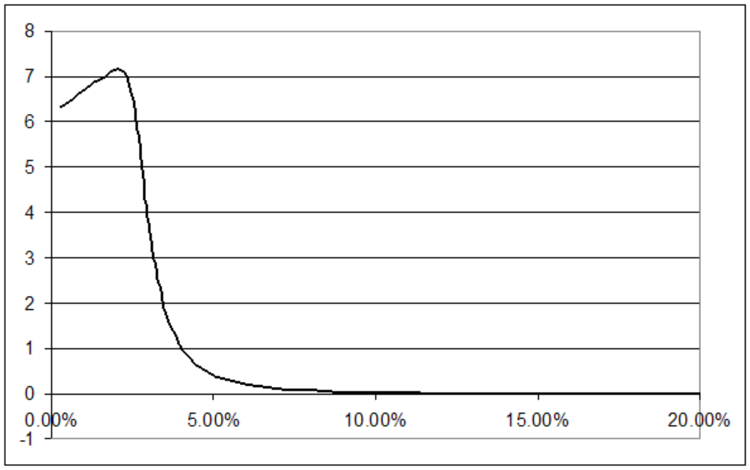

FIGURE 11. Delta of a 10Y Bermuda swaption (EUR denominated), with semi-annual exercise schedule, with respect to the 10Y swap rate. This is computed while holding fixed the volatility state variable and varying the strike level.

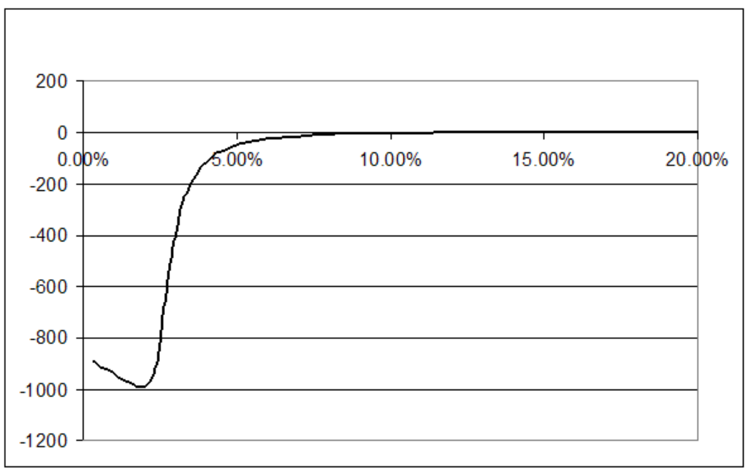

FIGURE 12. Gamma of a 10Y Bermuda swaption (EUR denominated), with semi-annual exercise schedule, with respect to the 10Y swap rate. This is computed while holding fixed the volatility state variable and varying the strike level.

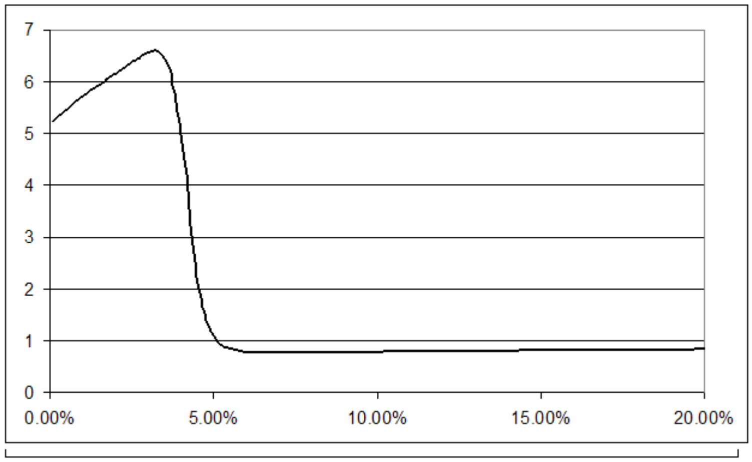

FIGURE 13. Delta of a 10Y callable CMS swap (EUR denominated), paying the 5Y swap rate with semi-annual exercise schedule, with respect to the 15Y swap rate . This is computed while holding fixed the volatility state variable and varying the strike level.

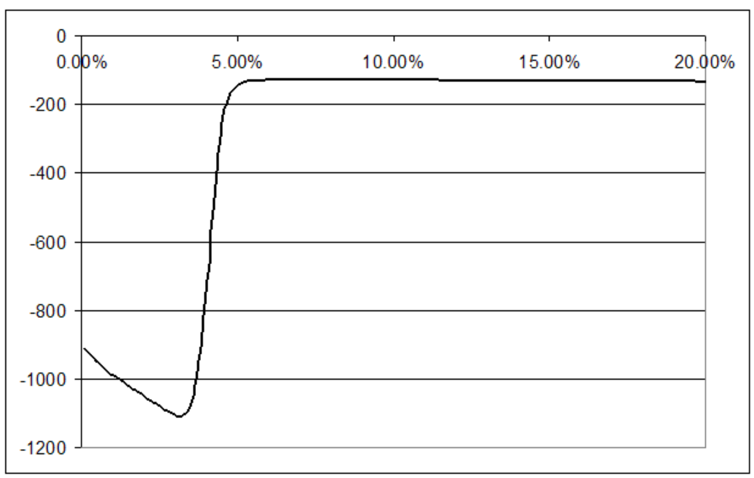

FIGURE 14. Gamma of a 10Y callable CMS swap (EUR denominated), paying the 5Y swap rate with semi-annual exercise schedule, with respect to the 15Y swap rate. This is computed while holding fixed the volatility state variable and varying the strike level.

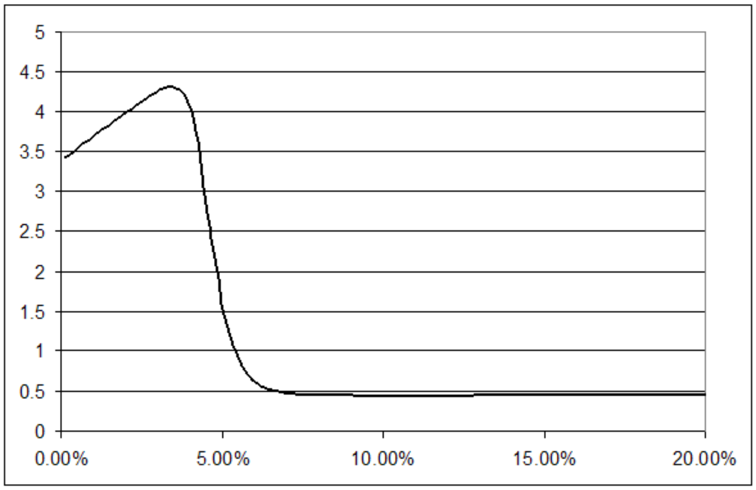

FIGURE 15. Vega of a 10Y callable CMS swap (EUR denominated), paying the 5Y swap rate with semi-annual exercise schedule, with respect to the 10Y into 5Y European swaption price. This is computed while holding fixed the short rate and varying the strike level.