BLACK-SCHOLES GOES HYPERGEOMETRIC

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ABSTRACT. We introduce a general pricing formula that extends Black-Scholes' and contains as particular cases most analytically solveable models in the literature, including the quadratic and the constant-elasticity-of-variance (CEV) models for European and barrier options. In addition, large families of new solutions can be found, containing as many as seven free parameters.

1. INTRODUCTION

It has been known since the seventies that Black-Scholes pricing formulas are a special case of more general families of pricing formulas with more than just the volatility as an adjustable parameter. The list of the classical extensions includes affine, quadratic and the constant-elasticity-of-variance models. These models admit up to three adjustable parameters and have found a variety of applications to solving pricing problems for equity, foreign exchange, interest rate and credit derivatives. In a series of working papers, the authors have recently developed new mathematical techniques that allow to go much further and to build several families of pricing formulas with up to seven adjustable parameters in the stationary, driftless case, and additional flexibility in the general time dependent case. The formulas extend to barrier options and have a similar structure as the Black-Scholes formulas, the most notable difference being that error functions (or cumulative normal distributions) are replaced by (confluent) hypergeometric functions, i.e. the special transcendental functions of applied mathematics and mathematical physics.

Let F denote a generic financial observable which we know is driftless. Examples could be the forward price of a stock or foreign currency under the forward measure, a LIBOR forward rate or a swap rate with the appropriate choice of numeraire asset. Black or Black-Scholes formulas are obtained by postulating that the time evolution of F obeys a stochastic differential equation of the form

$$dF_t = \sigma(F_t)dW_t \tag{1.1}$$

where $\sigma(F) = \sigma_1 F$ is linear. In this case, pricing formulas of calls and puts for both plain vanilla and barrier options can be written in exact analytical form in terms of the error function or cumulative normal distribution. Interestingly, quadratic volatility models with

$$\sigma(F) = \sigma_0 + \sigma_1 F + \sigma_2 F^2. \tag{1.2}$$

also allow for pricing formulas that reduce to the evaluation of an error function. The reason why error functions suffice also to express pricing formulas in the more general case of quadratic volatility models is that these models can all be reduced to a Wiener process by means of a simple measure change and variable transformation of the form

$$F_t = F(x_t), \tag{1.3}$$

where the underlying x_t follows:

$$dx_t = dW_t. \tag{1.4}$$

The formula (1.3) applies under a pricing measure where assets are valued in terms of a suitably defined numeraire $g_t = g(x,t)$. For a review of change of numeraire methods in pricing theory we refer to [1] and [2]. Both functions $F(x)$ and $g(x,t)$ can be derived explicitly for any choice of parameters $\sigma_0, \sigma_1, \sigma_2$.

It also turns out that quadratic volatility models are the only stationary, driftless models for which the combination of a non-linear transformation of the form (1.3) together with a change of numeraire

Date: August 6th, 2001.

C. A. was supported in part by the National Science and Engineering Council of Canada. We thank Dilip Madan, Stephan Lawi, Vadim Linetsky and Andrei Zabidonov for discussions. Remaining errors are our own.

reduces the problem to a Wiener process x_t as the one in (1.4). In [3], Carr, Lipton and Madan address the question of whether it is possible to relax the condition of stationarity and find more general processes with drift and volatility both dependent on calendar time as well as on F which still reduce to the Wiener process and they find that the general solution admits as many as eleven time dependent functions.

A related line of reasoning leading to extensions of the Black-Scholes formula starts from the observation that the CEV models with state dependent volatility specified as follows:

$$\sigma(F) = \sigma_0(F - \bar{F})^{1+\theta} \tag{1.5}$$

with constants θ and $\bar{F}$, reduces to the Bessel process

$$dx_t = \lambda_0 dt + \nu_0 \sqrt{x_t} dW_t. \tag{1.6}$$

by means of a non-linear transformation combined with a measure change. In [4], Lipton derives general reducibility conditions to the more general processes

$$dx_t = (\lambda_0 + \lambda_1 x_t)dt + \nu_0\, x_t^\beta dW. \tag{1.7}$$

which are solvable for $\beta = 0, 1, \frac{1}{2}$. The case $\beta = 1$ is the lognormal (or affine) model leading to the Black-Scholes formula. The two cases $\beta = 0, \frac{1}{2}$ correspond to well-known solvable short rate models, namely the Vasicek and the Cox-Ingerssol-Ross (CIR) models. In [5], [6], Albanese and Campolieti find a general solution to the reducibility conditions by Carr, Lipton and Madan for stationary, driftless processes. In this article, we summarize our findings by presenting the general solution formula, originally derived using reducibility conditions and illustrate its use in a few particular cases. Since the original derivation is somewhat lengthy, in the Appendix we give a streamlined verification of its validity and refer to our other papers for a constructive derivation.

2. General Pricing Formula

In this section, we derive a general pricing formula for the models which are solvable by the reduction method.

Assume that the state variable x has a drift $\lambda(x)$ for which one can find the pricing kernel for the process

$$dx_t = \lambda(x_t)dt + \nu(x_t)dW_t. \tag{2.1}$$

The pricing kernel is the function $u(x, t; x_0, t_0)$ which solves the forward Fokker-Plank equation in the first pair of arguments and the backward Black-Scholes equation in the second pair. The latter equation can be written as follows:

$$u_{t_0}(x, t; x_0, t_0) + \frac{\nu(x_0)^2}{2} u_{x_0x_0}(x, t; x_0, t_0) + \lambda(x_0)u_{x_0}(x, t; x_0, t_0) = 0 \tag{2.2}$$

for $t \geq t_0$, with final time condition at terminal time $t_0 = t$ given by $u(x, t; x_0, t) = \delta(x - x_0)$ (a Dirac delta function). Recall that the pricing kernel can be interpreted as the price of a limiting butterfly spread option and is related to the second derivative of a call option written on the state variable x with respect to the ”strike” x_0. The processes in (1.7) are examples of analytically solvable models for which one can compute the pricing kernel.

Solvable pricing models can be constructed starting from the price function of a generic European style option written on x, i.e. a solution $v(x, t)$ of the Black-Scholes equation in (2.2) with an arbitrary final time condition at $t = 0$. The Laplace transform of such a function

$$\hat{v}(x, \rho) = \int_0^\infty e^{\rho(t_0 - t)} v(x, t_0 - t)\, dt. \tag{2.3}$$

is usually referred to as “time-independent Green’s function” and satisfies a second order ordinary differential equation with Dirac delta function source term $\delta(x - x_0)$. Let us consider the homogeneous part of this equation as given by

$$-\rho\hat{v}(x, \rho) + \frac{\nu(x)^2}{2}\hat{v}_{xx}(x, \rho) + \lambda(x)\hat{v}_x(x, \rho) = 0. \tag{2.4}$$

We find that functions $\hat{v}(x,\rho)$ solving this equation can be taken as the elementary building blocks for the construction of solvable pricing models for the F space processes. Because of this reason, we name $\hat{v}(x,\rho)$ the "generating function".

Armed with a solution $\hat{v}(x,\rho)$, we define a volatility function $\sigma(F)$ and an invertible monotonic transformation $F = F(x)$ and its inverse $x = X(F)$ such that

$$\sigma(F) = \frac{\sigma_0 \nu(X(F)) \exp\left(-2\int^{X(F)} \frac{\lambda(s)ds}{\nu(s)^2}\right)}{\hat{v}(X(F),\rho)^2} \tag{2.5}$$

with arbitrary constant σ_0 and where

$$\frac{dx}{\nu(x)} = \pm\frac{dF}{\sigma(F)}. \tag{2.6}$$

The two signs correpond to either monotonic increasing or monotonic decreasing transformations. The freedom in choosing the sign gives rise to 2 families of solutions which are different in the general case. As we verify in the Appendix, the process

$$g_t = \frac{e^{\rho t}}{\hat{v}(x_t,\rho)} \tag{2.7}$$

can be regarded as a forward price process and under the measure with g as a numeraire the state variable x_t drifts at rate $\lambda(x)$. Hence, the pricing kernel $U(F,t;F_0,0)$ for the overlying forward price F at time t can be evaluated in closed form as the expected reward from a limit butterfly spread contract with delta function payoff

$$U(F,t;F_0,0) = E_0\left[\frac{g_t}{g_0}\delta(F(x_t)-F)\right] \tag{2.8}$$

conditional on the price having value F_0 at initial time $t=0$. Here, the expectation is computed assuming that g_t is the numeraire and that the state variable x drifts at rate $\lambda(x)$. The final formula for the pricing kernel in F space is related to the kernel in the underlying x space as follows:

$$U(F,t;F_0,0) = \frac{\nu(X(F))}{\sigma(F)}\frac{\hat{v}(X(F),\rho)}{\hat{v}(X(F_0),\rho)}e^{-\rho t}u(X(F),t;x(F_0),0). \tag{2.9}$$

A European call option written on the forward price F_0 at current time $t=0$, struck at K and maturing at time $t=T$ can be priced in this model by computing the following integral:

$$C(K,T;F_0) = e^{-\rho T}\int_{X(K)}^{\infty} dx \frac{\hat{v}(x,\rho)}{\hat{v}(X(F_0),\rho)}(F(x)-K)u(x,T;X(F_0),0). \tag{2.10}$$

Barrier and lookback options can be handled by modifying the underlying kernel in x-space to account for the appropriate boundary conditions. This is accomplished by means of either integral representations or eigenfunction expansion methods, i.e. Green's function methods that are standard in the theory of Sturm-Liouville equations. See the articles of Davydov and Linetsky [7] for a discussion in an option pricing context.

3. Four families of solvable models

The case $\beta = 1$ is the usual lognormal (or affine) model. Of interest here is the other two families with $\beta = 0, \frac{1}{2}$ in equation (1.7) giving rise to solvable models in case $\lambda_1 < 0$ which admit as a singular limit a family with different properties in the non mean-reverting limit $\lambda_1 = 0$. This provides four examples of our methodology to generate exactly solvable models.

If $\beta = 0$ and $\lambda_1 = 0$ we recover the Wiener process with constant drift, which is readily transformed into a driftless Wiener process and thus supports only quadratic volatility functions in F space, including the lognormal Black-Scholes model as a special subcase. If $\beta = 0$ and $\lambda_1 < 0$ then the kernel in x space can be shown to be given by

$$u(x,t;x_0,0) = \frac{2\sqrt{2\lambda_1\sinh(\lambda_1 t)}}{\sqrt{\pi}\nu_0(e^{2\lambda_1 t}-1)}\sinh\left(\frac{\bar{x}_0\bar{x}}{\nu_0^2\lambda_1\sinh(\lambda_1 t)}\right)\exp\left[\frac{\bar{x}^2+\bar{x}_0^2 e^{2\lambda_1 t}}{\lambda_1\nu_0^2(e^{2\lambda_1 t}-1)}+\frac{\lambda_1}{2}t\right] \tag{3.1}$$

where $\bar{x} \equiv \lambda_0 + \lambda_1 x$, $\bar{x}_0 \equiv \lambda_0 + \lambda_1 x_0$. The generating function solves a special case of the confluent hypergeometric equation (Hermite's equation) with general solution

$$\hat{v}(x,\rho) = q_1 M\left(-\frac{\rho}{2\lambda_1}, \frac{1}{2}, \frac{(\lambda_0+\lambda_1 x)^2}{(\lambda_1 \nu_0^2)^2} \right) + q_2(\lambda_0+\lambda_1 x) M\left(-\frac{\rho}{2\lambda_1} + \frac{1}{2}, \frac{1}{2}, \frac{(\lambda_0+\lambda_1 x)^2}{(\lambda_1 \nu_0^2)^2} \right) \tag{3.2}$$

with arbitrary constants q_1,q_2. Here $M(a,b,z)$ is Kummer's function[8], i.e. the confluent hypergeometric function that is regular at $z=0$. In this expression, one can count 6 free dimensionless parameters. For each choice of these parameters, there are two different families corresponding to the two different choices of the sign in equation (2.6).

If $\beta = \frac{1}{2}$ and $\lambda_1 = 0$, then the pricing kernel for the state variable is expressed in terms of modified Bessel functions as follows:

$$u(x,t;x_0,0) = \left(\frac{x}{x_0}\right)^{\frac{1}{2}\left(\frac{2\lambda_0}{\nu_0^2}-1\right)} \frac{e^{-2(x+x_0)/\nu_0^2 t}}{\nu_0^2 t/2} I_{\frac{2\lambda_0}{\nu_0^2}-1}\left(\frac{4\sqrt{x x_0}}{\nu_0^2 t}\right) . \tag{3.3}$$

The generating function is

$$\hat{v}(x,\rho) = x^{\frac{1}{2}\left(1-\frac{2\lambda_0}{\nu_0^2}\right)} \left[q_1 I_{\frac{2\lambda_0}{\nu_0^2}-1}\left(\sqrt{\frac{8\rho x}{\nu_0^2}}\right) + q_2 K_{\frac{2\lambda_0}{\nu_0^2}-1}\left(\sqrt{\frac{8\rho x}{\nu_0^2}}\right) \right] , \tag{3.4}$$

with arbitrary constants q_1,q_2. Here $I_\nu(z)$ is the modified Bessel function of order ν and $K_\nu(z)$ is the associated McDonalds function. In this case we obtain a dual family with 6 adjustable parameters.

The case $\beta = \frac{1}{2}$ and $\lambda_1 < 0$ is more general than the Vasicek case. Geometrically, the CIR process describes the stochastic dynamics of the radial distance of a point whose Cartesian coordinates follow an evolution given by the Vasicek model. This analogy applies only to integer dimensions, but analytic continuation in the dimension parameter allows one to gain one additional degree of freedom. The pricing kernel for the state variable x corresponds to that of the short rate CIR model, and can still be expressed in terms of modified Bessel functions as follows:

$$u(x,t;x_0,0) = c_t \left(\frac{x e^{-\lambda_1 t}}{x_0}\right)^{\frac{1}{2}\left(\frac{2\lambda_0}{\nu_0^2}-1\right)} \exp\left[-c_t(x_0 e^{\lambda_1 t} + x)\right] I_{\frac{2\lambda_0}{\nu_0^2}-1}\left(2c_t\sqrt{x x_0 e^{\lambda_1 t}}\right) , \tag{3.5}$$

where $c_t \equiv 2\lambda_1/(\nu_0^2(e^{\lambda_1 t}-1))$. For a derivation see [9]. The general solution of equation (2.4) reduces to Whittaker's equation and generating functions have the general form

$$\hat{v}(x,\rho) = x^{-\lambda_0/\nu_0^2} e^{-\lambda_1 x/\nu_0^2} \left[q_1 W_{k,m}\left(-\frac{2\lambda_1}{\nu_0^2}x\right) + q_2 M_{k,m}\left(-\frac{2\lambda_1}{\nu_0^2}x\right) \right] \tag{3.6}$$

for arbitrary constants q_1,q_2. Here $W_{k,m}(\cdot)$ and $M_{k,m}(\cdot)$ are Whittaker functions which can also be expressed in terms of confluent hypergeometric functions or in terms of Kummer functions.[8] This construction gives rise to a dual family with 7 free parameters where

$$k = \frac{\lambda_0}{\nu_0^2} + \frac{\rho}{\lambda_1} , \ m = \frac{\lambda_0}{\nu_0^2} - \frac{1}{2} . \tag{3.7}$$

The 7 parameter family which reduces to the CIR model has a local volatility function defined on either an interval or on a half line and behaves asymptotically as the CEV volatility on one hand and as a quadratic model on the other. This hibrid shape allows for a great deal of flexibility in reproducing observed volatility skews. In Figure 1 we show

Additional extensions are possible. For instance, one can apply a deterministic time change and still retain solvability. It is also possible to apply stochastic time changes and arrive to solvable extensions of the variance-gamma model, but we refer to forthcoming articles for a discussion of this and other related topics.

4. Recovering exact solutions in the literature

In this section we show that the known exact solutions in the literature, namely quadratic and CEV models, can all be rediscovered as particular cases of our general formula for the Bessel family where

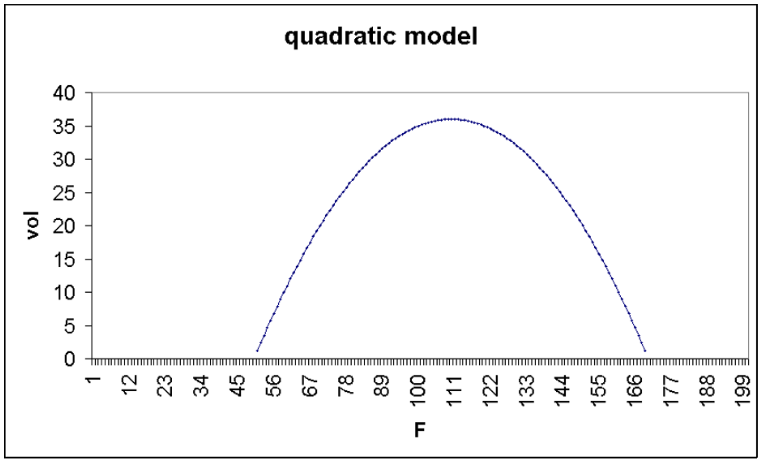

FIGURE 1. *Examples of local volatility functions for a quadratic model.*

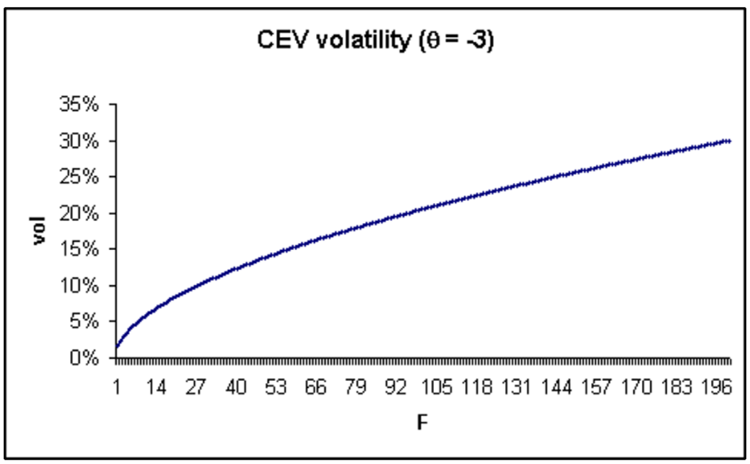

FIGURE 2. *Examples of local volatility unctions for the CEV model*

we make use of the above solutions to the underlying x space process with $\beta = \frac{1}{2}$, $\lambda_1 = 0$ and $\lambda \equiv \lambda_0$. Without loss of generality, we can fix $\nu_0 = 2$. Let's specialize further to the case where

$$F(x) = \bar{F} - a\frac{K_{\frac{\lambda}{2}-1}(\sqrt{2\rho x})}{I_{\frac{\lambda}{2}-1}(\sqrt{2\rho x})} \tag{4.1}$$

which leads to a process for the forward price F with volatility

$$\sigma(F) = \frac{a}{\sqrt{X(F)}\left[I_{\frac{\lambda}{2}-1}(\sqrt{2\rho X(F)})\right]^2}, \tag{4.2}$$

where $x = X(F)$ is the inverse of the function in equation (4.1). In this family, a and ρ are positive, $\bar{F}$ is arbitrary and $\lambda > 2$. The function $F(x)$ maps the half line $x \in [0, \infty)$ into $F \in (-\infty, \bar{F}]$, where $F(x)$ is a strictly monotonically increasing function with $dF(x)/dx = \sigma(F(x))/\nu(x)$. This solution region can be inverted so that $F \in [\bar{F}, \infty)$. This is accomplished by either replacing a by $-a$ in equation (4.1) or by applying a linear change of variables that maps F into $2\bar{F} - F$. In this special case, we make use of

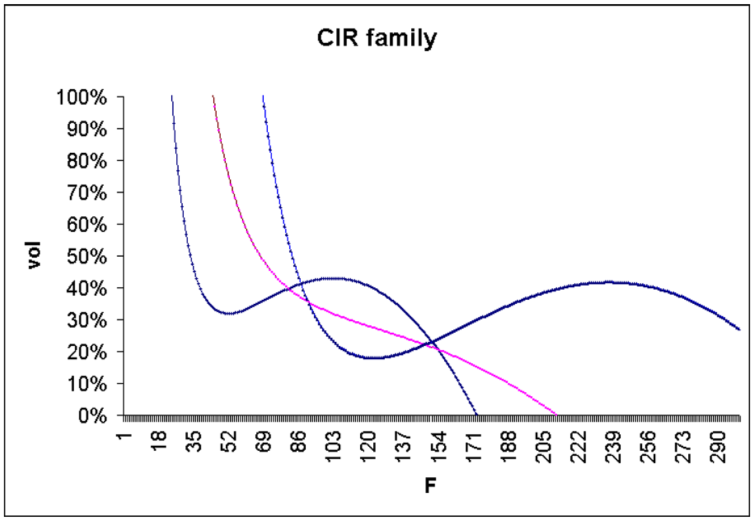

FIGURE 3. *Examples of local volatility for the CEV model*

the generating function in equation (3.4), with the choice $q_2 = 0$, and formula (2.9) reduces to

$$U(F,t;F_0,0) = \frac{e^{-\rho t-(X(F)+X(F_0))/2t}}{at}\frac{X(F)\left[I_{\frac{\lambda}{2}-1}(\sqrt{2\rho X(F)})\right]^3}{I_{\frac{\lambda}{2}-1}(\sqrt{2\rho X(F_0)})}I_{\frac{\lambda}{2}-1}\left(\frac{\sqrt{X(F)X(F_0)}}{t}\right). \tag{4.3}$$

We note that this density integrates exactly to unity in F space (i.e. no absorption).

4.1. **Quadratic volatility models.** Pricing kernels for quadratic volatility models are readily obtained as a subset of the above general family with the special choice of parameter $\lambda = 3$. After making the substitution $F \to 2\bar{F} - F$ and setting $a = (\bar{F} - \bar{\bar{F}})/\pi$ the transformation function $F(x)$ becomes

$$F(x) = \bar{F} + \frac{(\bar{F}-\bar{\bar{F}})}{\pi}\frac{K_{\frac{1}{2}}(\sigma_0\sqrt{x}/2)}{I_{\frac{1}{2}}(\sigma_0\sqrt{x}/2)} = \bar{F} + \frac{(\bar{F}-\bar{\bar{F}})}{\exp(\sigma_0\sqrt{x})-1} \tag{4.4}$$

where $\sigma_0 > 0$. Here, we assume that $\bar{F} > \bar{\bar{F}}$. The inverse transformation $X(F)$ is given by

$$X(F) = (1/\sigma_0^2)\log^2[1 + (\bar{F}-\bar{\bar{F}})/(F-\bar{F})], \tag{4.5}$$

and the volatility function $\sigma(F)$ is obtained by insertion into equation (4.2) while using the Bessel function of order $\frac{1}{2}$,

$$\sigma(F) = \frac{\sigma_0}{(\bar{F}-\bar{\bar{F}})}(F-\bar{F})(F-\bar{\bar{F}}). \tag{4.6}$$

Inserting the expression (4.5) into equation (4.3), one obtains the pricing kernel

$$U(F,t;F_0,0) = \frac{2e^{-\sigma_0^2 t/8}}{\sigma(F)\sqrt{2\pi t}}\sqrt{\frac{(F_0-\bar{F})(F_0-\bar{\bar{F}})}{(F-\bar{F})(F-\bar{\bar{F}})}}e^{-(\phi(F)^2+\phi(F_0)^2)/2\sigma_0^2 t}\sinh\left(\frac{\phi(F_0)\phi(F)}{\sigma_0^2 t}\right) \tag{4.7}$$

where $\phi(F) \equiv \log((F-\bar{\bar{F}})/(F-\bar{F}))$. In the special case of a volatility function with a double root, i.e.

$$\sigma(F) = \sigma_0(F-\bar{F})^2 \tag{4.8}$$

the pricing kernel is computed by taking the limit as $\bar{\bar{F}} \to \bar{F}$, and one finds

$$\begin{aligned} U(F,t;F_0,0) = \frac{1}{\sigma_0\sqrt{2\pi t}}\frac{(F_0-\bar{F})}{(F-\bar{F})^3}\Big[& e^{-\left((F-\bar{F})^{-1}-(F_0-\bar{F})^{-1}\right)^2/2\sigma_0^2 t} \\ & -e^{-\left((F-\bar{F})^{-1}+(F_0-\bar{F})^{-1}\right)^2/2\sigma_0^2 t}\Big]. \end{aligned} \tag{4.9}$$

4.2. **Lognormal models.** The pricing kernel for the log-normal Black-Scholes model with $\sigma(F) = \sigma_0 F$ is a particular case of the above formula for the quadratic model. The derivative with respect to F of the quadratic volatility function in (4.6), evaluated at $F = \bar{F}$, is σ_0. Taking the limit $\bar{\bar{F}} \to -\infty$ (or $\bar{\bar{F}} << \bar{F}$), while holding the other parameters fixed, one obtains $\sigma(F) = \sigma_0(F - \bar{F})$. The pricing kernel in (4.7) gives the kernel for the log-normal model in the limit $\bar{\bar{F}} \to -\infty$, i.e.

$$U(F, t; F_0, 0) = \frac{1}{(F - \bar{F})\sigma_0\sqrt{2\pi t}} \exp\left[- \left(\log((F_0 - \bar{F})/(F - \bar{F})) - \frac{\sigma_0^2}{2} t \right)^2 \Big/ 2\sigma_0^2 t \right]. \tag{4.10}$$

4.3. **CEV model.** The constant-elasticity-of-variance (CEV) model is recovered in the limiting case as $\rho \to 0$. Assume $\lambda > 2$ and let $\theta > 0$ be defined so that $\lambda = \theta^{-1} + 2$. The transformation $F = F(x)$

$$F(x) = \bar{F} + (\sigma_0^2 x)^{-(2\theta)^{-1}} \tag{4.11}$$

has inverse $x = X(F)$ given by

$$X(F) = \sigma_0^{-2}(F - \bar{F})^{-2\theta}, \tag{4.12}$$

for any constant $\bar{F}$. The volatility function for this model is

$$\sigma(F) = \frac{\sigma_0}{|\theta|}(F - \bar{F})^{1+\theta}. \tag{4.13}$$

In the limit $\rho \to 0$, the Laplace transform $\hat{v}(X(F), 0) = 1$, which implies that the numeraire change is trivial in this case. The pricing kernel can be evaluated by substitution into the general formula (2.9), and after collecting terms, it turns out to be

$$\begin{aligned} U(F, t; F_0, 0) = \frac{|\theta|}{\sigma_0^2 t} \frac{(F_0 - \bar{F})^{\frac{1}{2}}}{(F - \bar{F})^{\frac{3}{2}+2\theta}} e^{-\left((F-\bar{F})^{-2\theta} + (F_0 - \bar{F})^{-2\theta}\right)/2\sigma_0^2 t} \\ I_{\frac{1}{2\theta}}\left(\frac{\left((F - \bar{F})(F_0 - \bar{F})\right)^{-\theta}}{\sigma_0^2 t} \right). \end{aligned} \tag{4.14}$$

This formula was derived in the case $\theta > 0$, for which the limiting value $F = \bar{F}$ is not attained and the density is easily shown to integrate to unity (i.e. no absorption occurs and the density also vanishes at the endpoint $F = \bar{F}$). We note that the same formula solves the forward pricing equation for $\theta < 0$, leading to the same Bessel equation of order $\pm(2\theta)^{-1}$. In the range $\theta < 0$, however, the properties of the above pricing kernel are generally more subtle. In particular, one can show that the density integrates to unity for all values $\theta < -1/2$, hence no absorption occurs for $\theta \in (-\infty, -1/2)$. The boundary conditions for the density can be shown to be vanishing at $F = \bar{F}$ (i.e. paths do not attain the lower endpoint) for all $\theta < -1$. In contrast, for $\theta \in (-1, -1/2)$ the density becomes singular at the lower endpoint $F = \bar{F}$ (hence this corresponds to the case that the density has an integrable singularity for which paths can also attain the lower endpoint, but are not absorbed). For the special case of $\theta = -1/2$ the formula gives rise to absorption. [Note that only for the range $\theta \in (-1/2, 0)$ the above pricing kernel is not useful since it gives rise to a density that has a non-integrable singularity at $F = \bar{F}$. In this case, however, another solution that is integrable is obtained by only replacing the order $(2\theta)^{-1}$ by $-(2\theta)^{-1}$ in the Bessel function. The latter solution for the density does not integrate to unity and hence gives rise to absorption which can be of use to price options in a credit setting.] The special case of $\theta = -1$ gives a nonzero constant value at the lower endpoint, and recovers the Wiener process with reflection and no absorption on the interval $[\bar{F}, \infty)$ with

$$U(F, t; F_0, 0) = \frac{1}{\sigma_0\sqrt{2\pi t}} \left(e^{-(F-F_0)^2/2\sigma_0^2 t} + e^{-(F+F_0-2\bar{F})^2/2\sigma_0^2 t} \right). \tag{4.15}$$

5. Barrier Options

The original motivation of two of us, C.A. and G.C., as we engaged in this project, was to streamline the derivation of pricing formulas for barrier options for our class of financial engineering master students. The general expression for the pricing kernel in this article gives in fact a simple and straightforward derivation of pricing formulas for barrier options, by allowing one to reduce to the case of a standard Brownian motion in x space.

Consider as an example a down-and-out option with barrier at $F = H$ within the Black-Scholes model with $\sigma(F) = \sigma_0 F$. This reduces to the driftless Wiener process with volatility $\nu(x) = \sqrt{2}$, by means of the transformation where

$$x = X(F) = (\sqrt{2}/\sigma_0)\log F \tag{5.1}$$

with inverse $F = F(x) = e^{\sigma_0 x/\sqrt{2}}$. Specializing equation (2.9) gives

$$U(F, t; F_0, 0) = \frac{\sqrt{2}}{\sigma_0 F}\exp\left[\frac{1}{2}\log(F_0/F) - \frac{\sigma_0^2}{8}t\right]u(X(F), t; X(F_0), 0)\,. \tag{5.2}$$

The region $x \in (-\infty, \infty)$ maps into $F \in (0, \infty)$. A barrier located at $F = H$ corresponds to $H = F(x_H) = e^{\sigma_0 x_H/\sqrt{2}}$, so $x_H = X(H) = (\sqrt{2}/\sigma_0)\log H$. The upper region $F \in [H, \infty)$ maps into $x \in [x_H, \infty)$. The x-space kernel with absorbing boundary condition at $x = x_H$ is obtained by the method of images, as

$$u(x, t; x_0, 0) = \frac{1}{\sqrt{4\pi t}}\big(e^{-(x-x_0)^2/4t} - e^{-(x+x_0-2x_H)^2/4t}\big). \tag{5.3}$$

Inserting this kernel into the general pricing formula in (5.2), immediately gives the pricing kernel in F space:

$$U^H(F, t; F_0, 0) = U(F, t; F_0, 0)\Big[1 - \exp\Big[-\frac{\log(F/H)\log(F_0/H)}{\sigma_0^2 t/2}\Big]\Big] \tag{5.4}$$

where $U(F, t; F_0, 0)$ is the barrier-free pricing kernel

$$U(F, t; F_0, 0) = \frac{1}{\sigma_0 F\sqrt{2\pi t}}\exp\left[-\left(\log(F_0/F) - \frac{\sigma_0^2}{2}t\right)^2/2\sigma_0^2 t\right]. \tag{5.5}$$

A down-and-out call maturing at time T and struck at $K > H$, has price at time $t = 0$ given by the integral

$$C^{DO}(F_0, K, T) = \int_H^\infty dF\, U^H(F, T; F_0, 0)(F - K)_+\,. \tag{5.6}$$

where F_0 is the current forward of maturity T. This integral can be evauated in terms of cumulative normal distribution functions as follows:

$$\begin{aligned} C^{DO}(F_0, K, T) = F_0 N(d_1(F_0/K)) \quad &- K N(d_2(F_0/K)) - H N(d_1(H^2/F_0 K)) \\ &+ (K F_0/H) N(d_2(H^2/F_0 K)) \end{aligned} \tag{5.7}$$

where

$$d_1(x) = \frac{\log x + \frac{1}{2}\sigma_0^2 T}{\sigma_0\sqrt{T}} \tag{5.8}$$

and $d_2(x) = d_1(x) - \sigma_0\sqrt{T}$.

In the more general case of the other solvable models such as the dual family of 7 parameter models reducible to the CIR process, one can also obtain analytic closed form solutions for a variety of exotic payoffs, including simple barrier options. Based on our general results, the derivation of pricing formulas is straightforward and will be presented elsewhere.

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6. APPENDIX

In this appendix we derive the main formula in section 2. Consider the situation of a generic pricing measure whereby the process for x_t obeys the equation

$$dx_t = \mu(x_t)dt + \nu(x_t)dW_t \tag{6.1}$$

for some drift $\mu(x)$. Then the process g_t defined in (2.7) satisfies the equation

$$dg = \left(\rho - \mu\frac{\hat{u}_x}{\hat{u}} + \nu^2\left[\left(\frac{\hat{u}_x}{\hat{u}}\right)^2 - \frac{1}{2}\frac{\hat{u}_{xx}}{\hat{u}}\right]\right)gdt + \sigma^g g dW \tag{6.2}$$

where

$$\sigma^g = -\frac{\hat{u}_x \nu}{\hat{u}} \tag{6.3}$$

is defined as the log-normal volatility of g. Note that in this appendix we write the function σ^g (with the superscript g) to mean the volatility function of the underlying g_t. This volatility is of course a different function from the volatility function $\sigma(F)$ of the forward price. Note also that here the subscript variable x stands for differentiation and subscript xx stands for double differentiation of the function with respect to x, respectively. Substituting equation (2.4)

$$\frac{\nu^2}{2}\hat{u}_{xx} = \rho\hat{u} - \lambda\hat{u}_x \tag{6.4}$$

into this equation, we find that

$$\frac{dg}{g} = \left(\frac{\mu - \lambda}{\nu}\sigma^g + (\sigma^g)^2\right)dt + \sigma^g dW. \tag{6.5}$$

To demonstrate that g defines a forward price process, consider this equation in the original forward measure where the forward price F follows a martingale process. In this case, using Ito's Lemma on the mapping $x_t = X(F_t)$ and equation 1.1 we arrive at an SDE of the form in equation (6.1) with drift given by

$$\mu(x) = \frac{\sigma(F)^2}{2}\frac{d}{dF}\frac{dX(F)}{dF} = \frac{\sigma(F)^2}{2}\frac{d}{dF}\frac{\nu(x)}{\sigma(F)} \tag{6.6}$$

where the monotonic mapping $dX(F)/dF = \nu(x)/\sigma(F)$ has been used. Using the chain rule for differentiation, and expressing all functions in terms of x, we then have

$$\mu(x) = \frac{\sigma\nu}{2}\frac{d}{dx}\left(\frac{\nu}{\sigma}\right) = \frac{\nu}{2}\left[\nu_x - \frac{\nu}{\sigma}\sigma_x\right], \tag{6.7}$$

where $\sigma \equiv \sigma(F(x))$ is the volatility function for the forward price F. Hence, by substitution the drift of g in the forward measure given by equation 6.5 is

$$\frac{\mu - \lambda}{\nu}\sigma^g + (\sigma^g)^2 = \left[\lambda + \frac{\nu^2}{2}\frac{\sigma_x}{\sigma} - \frac{1}{2}\nu\nu_x\right]\frac{\hat{u}_x}{\hat{u}} + \nu^2\left(\frac{\hat{u}_x}{\hat{u}}\right)^2. \tag{6.8}$$

By using the volatility function of the forward price defined in terms of the nonlinear transformation in the x variable, namely

$$\sigma = \frac{\sigma_0\nu(x)e^{-2\int^x \frac{\lambda(y)dy}{\nu(y)^2}}}{\hat{u}(x,\rho)^2}, \tag{6.9}$$

we find

$$\frac{\sigma_x}{\sigma} = \frac{\nu_x}{\nu} - \frac{2\lambda}{\nu^2} - \frac{2\hat{u}_x}{\hat{u}}. \tag{6.10}$$

Substituting into (6.8), we find that the drift of g under the forward measure vanishes. Hence, g describes a forward price process, as stated.

Next, consider equation (6.5) again, but this time under the measure having the forward g as numeraire. Under this pricing measure the price of risk is σ^g and

$$dg = (\sigma^g)^2 g dt + \sigma^g g dW. \tag{6.11}$$

Comparison with equation (6.5) shows that under this measure the drift of the underlying process x_t is λ, as stated. This implies that the representation (2.9) for the pricing kernel is correct and completes the proof.

We refer the reader interested in gaining further insight into this derivation to our article [6]. There we provide a proof of this result, which is more elaborate and fully constructive, and is not based on the above stochastic analysis argument.

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